Closed and Exact Differential Forms-HW Problems

1. Let F be a vector field on $\mathbb{R}^3$. Thus $F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$. Define the following differential forms based on F:
   $\omega_1(F) = F_1 dx + F_2 dy + F_3 dz$ and
   $\omega_2(F) = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$.
   a. Suppose that $F = grad(f)$. Show that $\omega_1(gradf) = df$.
   b. Show for any smooth vector field F, $d(\omega_1(F)) = \omega_2(curl(F))$.
   c. Show for any smooth vector field F, $d(\omega_2(F)) = Div(F) dx \wedge dy \wedge dz$.
   d. Use the earlier parts to show that $curl\big(grad(f)\big) = 0$ and $div\big(curl(F)\big) = 0$.
   e. If F is a smooth vector field on a convex set A in $\mathbb{R}^3$and $curl(F) = 0$, show that $F = grad(f)$ for some smooth function on A. Similarly, show if $div(F) = 0$, then $F = curl(G)$ for some smooth vector field G.

2. Let ω and η be k and l forms on $\mathbb{R}^n$.
   a. Show if ω and η are closed then so is $\omega \wedge \eta$.
   b. Show if ω and η are exact then so is $\omega \wedge \eta$.

3. Prove that any $n + 1$ form on $\mathbb{R}^n$ is 0.

4. Let $\omega = (1 + ye^{xy})dx + (2y + xe^{xy})dy$ be a 1-form on $\mathbb{R}^2$.
   a. Show $d\omega = 0$.
   b. Find all functions $f(x, y)$ such that $\omega = df$. Notice that $f(x, y)$ is a solution to the differential equation:

$$(1 + ye^{xy}) + (2y + xe^{xy})\left(\frac{dy}{dx}\right) = 0.$$