Stokes' Theorem

Recall that the vector form of Green's theorem says:

Vector Form of Green's Theorem: Let $D \subset \mathbb{R}^2$ be a region to which Green's theorem applies. Let ∂D be its positively oriented (ie counterclockwise) boundary, and let $\vec{F} = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ be a C^1 vector field on D then

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (curl\, \vec{F}) \cdot \vec{k} dA = \iint_D (\nabla \times \vec{F}) \cdot \vec{k} dA.$$

Notice that since Green's theorem applies to regions in the xy-plane (ie a surface that lies in the xy-plane), the $\vec{k}$ that appears on the right hand side of the formula is actually the unit normal, $\vec{n}$, to the surface D. Also, since we are dealing with a surface in the xy-plane, the "dA" is the same as "dS". Thus we could write the formula as:

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (curl\, \vec{F}) \cdot \vec{n} dS = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS.$$

Now if the surface is parametrized we can write:

$$\vec{n} = \frac{\vec{T}_u \times \vec{T}_v}{|\vec{T}_u \times \vec{T}_v|}, \quad dS = \left|\vec{T}_u \times \vec{T}_v\right| dudv, \quad d\vec{S} = (\vec{T}_u \times \vec{T}_v) dudv$$

So $\vec{n} dS = d\vec{S}$.

So we could write Green's theorem as:

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (curl\, \vec{F}) \cdot d\vec{S} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$$

This is exactly what the conclusion to Stokes' theorem is. The difference is that the surface, S, in Stokes' theorem is a surface in $\mathbb{R}^3$, not in $\mathbb{R}^2$.

Stokes' Theorem (for graphs, $z = f(x, y)$): Let S be the oriented surface defined by a C^2 function $z = f(x, y)$, where $(x, y) \in D$, a region to which Green's theorem applies, and let $\vec{F}$ be a C^1 vector field on S. Then if ∂S denotes the oriented boundary curve of S, we have:

$$\int_{\partial S} \vec{\boldsymbol{F}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}} = \iint_S \left(\boldsymbol{curl}\,\vec{\boldsymbol{F}}\right) \cdot \boldsymbol{d}\vec{\boldsymbol{S}} = \iint_S \left(\nabla \times \vec{\boldsymbol{F}}\right) \cdot \boldsymbol{d}\vec{\boldsymbol{S}}.$$

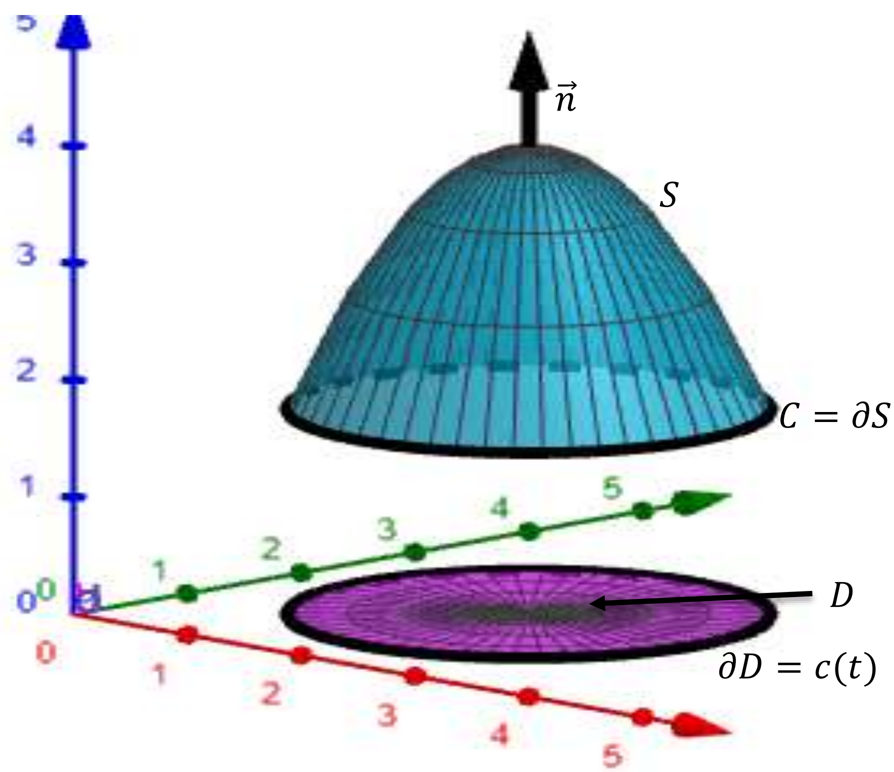
Note: As mentioned above, since $\vec{n}dS = d\vec{S}$ we could rewrite this as:

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \left(curl\,\vec{F}\right) \cdot \vec{n}dS = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}dS \quad \text{or}$$

$$\int_{\partial S} \vec{F} \cdot \vec{T}ds = \iint_S \left(curl\,\vec{F}\right) \cdot \vec{n}dS = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}dS$$

i.e., The integral of the normal component of the curl($\vec{F}$) over the surface S is equal to the integral of the tangential component of $\vec{F}$ around ∂S.

The idea of the proof is to reduce Stokes' theorem to an application of Green's theorem. This can be done by using the fact that if $f: D \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ then $f(\partial D) = \partial S$. So if the boundary of D is given by the curve $\vec{c}(t) = < x(t), y(t) >$, then ∂S is given by the curve $\vec{C}(t) = < x(t), y(t), f\big(x(t), y(t)\big) >$.



Thus we can write:

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{\partial S} (F_1 dx + F_2 dy + F_3 dz); \quad \text{where}$$

$$dx = \frac{dx}{dt} dt, \quad dy = \frac{dy}{dt} dt, \quad dz = \frac{dz}{dt} dt = \left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right) dt$$

Plugging dx, dy, dz into the integral and rearranging terms we get

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$= \int_{\partial D} \left(F_1 + F_3 \frac{\partial z}{\partial x}\right) dx + \left(F_2 + F_3 \frac{\partial z}{\partial y}\right) dy$$

If we now apply Green's theorem to this integral we get a messy integral over D.

If we now calculate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ using the fact that if $z = f(x, y)$ then

$$\iint_S \vec{G} \cdot d\vec{S} = \iint_D (-G_1 f_x - G_2 f_y + G_3)\, dydx$$

and using expression $\vec{G} = \nabla \times \vec{F}$, the RHS of the integral in the line above is actually equal to the messy integral we just got from Green's theorem.

Ex. Let $\vec{F} = (e^x + z)\vec{\imath} + (cosy)\vec{\jmath} + x\vec{k}$. Show that the integral of $\vec{F}$ around an oriented simple closed curve c that is the boundary of a C^2 surface S (where S is of the form $z = f(x, y)$) is 0.

By Stokes' theorem we know: $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

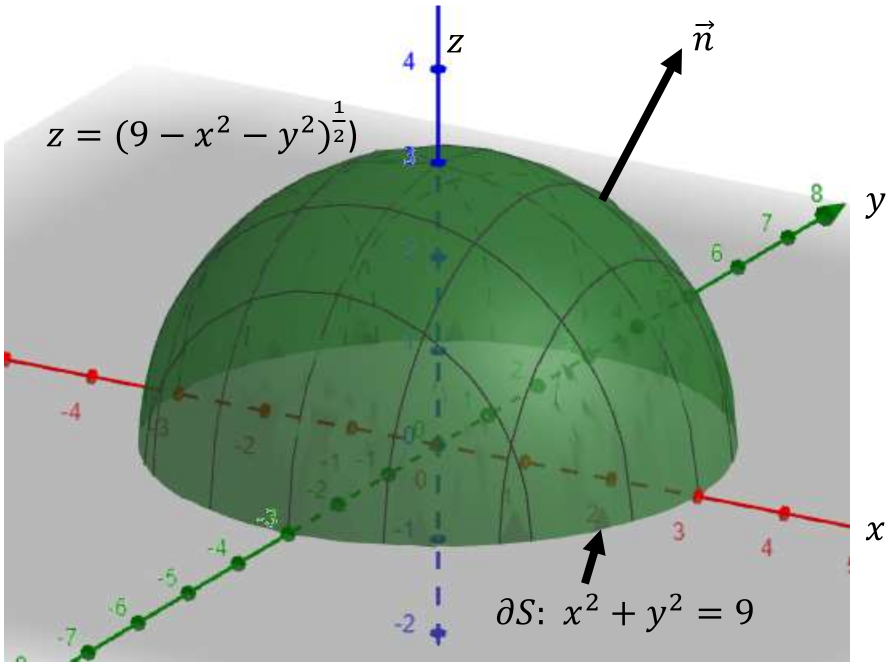
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x + z & cosy & x \end{vmatrix} = \vec{0}$$

So we have $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \vec{0} \cdot d\vec{S} = 0$.

Ex. Verify Stokes' theorem for $\vec{F} = (z^2)\vec{\imath} + (x)\vec{\jmath} + (y^2)\vec{k}$ and the surface

$S = \{(x, y, z): x^2 + y^2 + z^2 = 9;\ \ z \geq 0\}$ (oriented as the graph

$z = (9 - x^2 - y^2)^{\frac{1}{2}})$ with $\partial S = \{(x, y): x^2 + y^2 = 9\}$.



We need to evaluate each side of: $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

LHS: ∂S is just a circle of radius 3 in the xy-plane.

$\vec{c}(t) =< 3cost, 3sint, 0 >; \quad 0 \leq t \leq 2\pi$

$\vec{c}'(t) =< -3sint, 3cost, 0 >;$

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} < 0^2, 3cost, 9sin^2 t > \cdot < -3sint, 3cost, 0 > dt$$

$$= \int_0^{2\pi} (9cos^2 t)dt = 9\int_0^{2\pi} (\frac{1}{2} + \frac{cos2t}{2})dt = 9\pi.$$

RHS: We have to calculate $\nabla \times \vec{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x & y^2 \end{vmatrix} = (2y)\vec{\imath} + (2z)\vec{\jmath} + \vec{k}.$$

Since the surface can be given as $z = f(x, y)$, ie $z = (9 - x^2 - y^2)^{\frac{1}{2}}$, we can either remember the formula:

$\iint_S \vec{G} \cdot d\vec{S} = \iint_D (-(G_1)z_x - (G_2)z_y + G_3)\, dxdy$ where $\vec{G} = \nabla \times \vec{F}$ or

rederive it by letting:

$\vec{\Phi}(x, y) =< x, y, z = (9 - x^2 - y^2)^{\frac{1}{2}} >$ and calculating $\vec{T}_x \times \vec{T}_y$.

In this example we will calculate $\vec{T}_x \times \vec{T}_y$.

$$\vec{\Phi}(x, y) =< x, y, (9 - x^2 - y^2)^{\frac{1}{2}} >$$

$$\vec{T}_x =< 1, 0, \frac{-x}{(9-x^2-y^2)^{\frac{1}{2}}} >; \qquad \vec{T}_y =< 0, 1, \frac{-y}{(9-x^2-y^2)^{\frac{1}{2}}} >$$

$$\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & \frac{-x}{(9-x^2-y^2)^{\frac{1}{2}}} \\ 0 & 1 & \frac{-y}{(9-x^2-y^2)^{\frac{1}{2}}} \end{vmatrix} = \frac{x}{(9-x^2-y^2)^{\frac{1}{2}}}\vec{\imath} + \frac{y}{(9-x^2-y^2)^{\frac{1}{2}}}\vec{\jmath} + \vec{k}$$

$$(\nabla \times \vec{F}) \cdot (\vec{T}_x \times \vec{T}_y)$$

$$= ((2y)\vec{\imath} + (2z)\vec{\jmath} + \vec{k})\cdot (\frac{x}{(9-x^2-y^2)^{\frac{1}{2}}}\vec{\imath} + \frac{y}{(9-x^2-y^2)^{\frac{1}{2}}}\vec{\jmath} + \vec{k})$$

$$= \frac{2xy}{(9-x^2-y^2)^{\frac{1}{2}}} + \frac{2yz}{(9-x^2-y^2)^{\frac{1}{2}}} + 1$$

$$= \frac{2xy}{(9-x^2-y^2)^{\frac{1}{2}}} + 2y + 1.$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{(x^2+y^2\leq 9)} \left(\frac{2xy}{(9 - x^2 - y^2)^{\frac{1}{2}}} + 2y + 1 \right) dxdy.$$

Now change to polar coordinates:

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} (\frac{2r^2 cos\theta sin\theta}{\sqrt{9-r^2}} + rsin\theta + 1)(r)drd\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} (\frac{2r^3 cos\theta sin\theta}{\sqrt{9-r^2}} + r^2 sin\theta + r)drd\theta$$

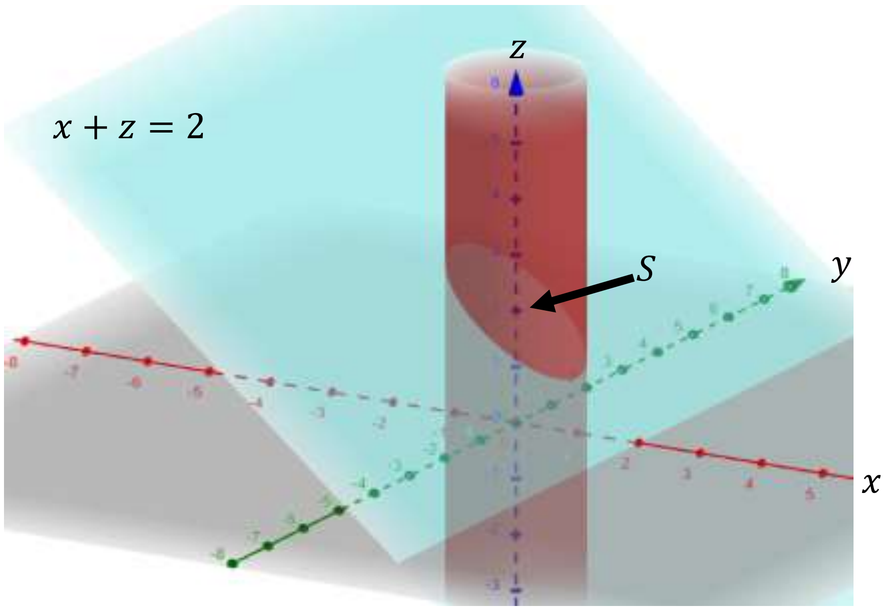
$$= 2\int_0^{2\pi} cos\theta sin\theta d\theta \int_0^3 \frac{r^3}{\sqrt{9-r^2}} dr + 2\int_0^{2\pi} sin\theta d\theta \int_0^3 r^2\, dr + \int_0^{2\pi}\int_0^3 rdrd\theta.$$

If we let $u = sin\theta$ in the first integral we will see that $\int_0^{2\pi} cos\theta sin\theta d\theta = 0$.

And since $\int_0^{2\pi} sin\theta d\theta = 0$ in the middel integral we also get 0:

$$= \ 0 + \ 0 + \ area\ of\ disk\ or\ radius\ 3 = 9\pi$$

Ex. Verify Stokes' theorem for $\vec{F} = (-y^2)\vec{\imath} + (x)\vec{\jmath} + (z^2)\vec{k}$ and S is the intersection of the solid cylinder $x^2 + y^2 \leq 1$ and the plane $x + z = 2$ (∂S is oriented counterclockwise).



We need to show: $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

RHS: First calculate $\nabla \times \vec{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\vec{k}.$$

Since the surface can be given as $z = f(x, y)$, ie $z = 2 - x$, we can either remember the formula:

$\iint_S \vec{G} \cdot d\vec{S} = \iint_D (-(G_1)z_x - (G_2)z_y + G_3)\, dxdy$ where $\vec{G} = \nabla \times \vec{F}$ or rederive it by letting

$\vec{\Phi}(x, y) =< x, y, 2 - x >$ and calculating $\vec{T}_x \times \vec{T}_y$.

In this example we will use $\iint_S \vec{G} \cdot d\vec{S} = \iint_D (-(G_1)z_x - (G_2)z_y + G_3)\, dxdy$.

Since $\vec{G} = \nabla \times \vec{F} = (1 + 2y)\vec{k}$, G_1 and G_2 are 0 and $G_3 = 1 + 2y$.

$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{x^2+y^2\leq 1}(1 + 2y)dxdy$.

Since we are integrating over a disk, change to polar coordinates:

$= \int_0^{2\pi}\int_0^1(1 + 2rsin\theta)rdrd\theta = \int_0^{2\pi}\int_0^1(r + 2r^2sin\theta)drd\theta$

$= \int_0^{2\pi}\frac{1}{2}r^2 + \frac{2}{3}r^3sin\theta\Big|_0^1 d\theta = \int_0^{2\pi}(\frac{1}{2} + \frac{2}{3}sin\theta)d\theta = \pi$.

LHS: To calculate $\int_{\partial S}\vec{F} \cdot d\vec{s}$ we need to parametrize ∂S.

The boundary curve is the intersection of $x^2 + y^2 = 1$ and $z = 2 - x$.

$x = cost, \qquad y = sint, \qquad z = 2 - x = 2 - cost, \qquad 0 \leq t \leq 2\pi.$

$\vec{c}(t) =< cost, sint, 2 - cost >; \quad 0 \leq t \leq 2\pi$

$\vec{c}'(t) =< -sint,\ cost,\ sint >;$

$\int_{\partial S}\vec{F} \cdot d\vec{s} = \int_0^{2\pi} < -sin^2t, cost, (2 - cost)^2 > \cdot < -sint, cost, sint > dt$

$= \int_0^{2\pi}[sin^3t + cos^2 t + sint(2 - cost)^2]dt$

$= \int_0^{2\pi}\sin^3 tdt + \int_0^{2\pi}\cos^2 tdt + \int_0^{2\pi}sint(2 - cost)^2dt$

$= \int_0^{2\pi}sint(1 - \cos^2 t)dt + \int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}cos2t\right)dt + \int_0^{2\pi}sint(2 - cost)^2dt.$

To evaluate the first integral let $u = cost,\ du = -sintdt$. To evaluate the 3rd integral let $u = 2 - cost,\ du = sintdt$.

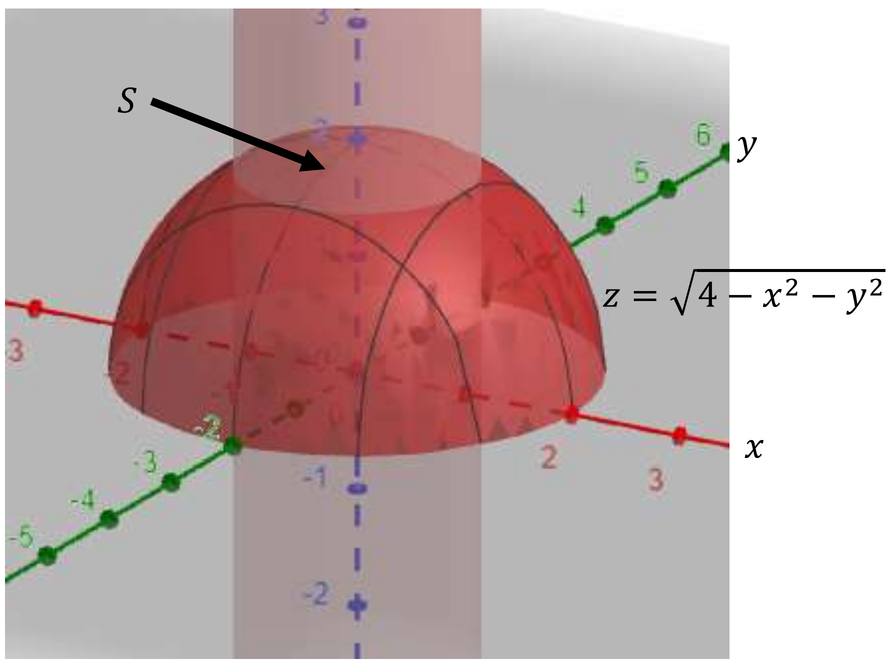
$$= -(-cost + \tfrac{1}{3}\cos^3 t)\Big|_0^{2\pi} + (\tfrac{1}{2}t + \tfrac{1}{4}sin2t\Big|_0^{2\pi} + \tfrac{1}{3}(2 - cost)^3\Big|_0^{2\pi}$$

$$= \pi.$$

Ex. Use Stokes' theorem to calculate $\iint_S\ curl(\vec{F}) \cdot d\vec{S}$ where

$\vec{F}(x, y, z) = (xz)\vec{\imath} + (yz)\vec{\jmath} + (xy)\vec{k}$ and S is the part of the sphere

$x^2 + y^2 + z^2 = 4$ that lies inside $x^2 + y^2 = 1$ and above the xy-plane.



$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S\ (curl\,\vec{F}) \cdot d\vec{S}$

∂S is the intersection of $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Substituting the second equation in the first we get:

$1 + z^2 = 4$, or $\ z = \pm\sqrt{3}$, but since $z \geq 0$, $\ z = \sqrt{3}$.

Thus the curve of intersection can be given by:

$x = cost, \quad y = sint, \quad z = \sqrt{3}$

$\vec{c}(t) =< cost, sint, \sqrt{3} >; \quad 0 \leq t \leq 2\pi.$

$\vec{c}'(t) =< -sint, cost, 0 >;$

$\vec{F}(\vec{c}(t)) =< \sqrt{3}cost, \sqrt{3}sint, costsint >.$

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} < \sqrt{3}cost, \sqrt{3}sint, costsint > \cdot < -sint, cost, 0 > dt$$

$$= \int_0^{2\pi} 0dt = 0$$

So $\iint_S (curl\,\vec{F}) \cdot d\vec{S}$ =0.

Notice that any smooth surface which has the same boundary, c, will give the same integral value of , ie if $\partial S_1 = \partial S_2$, where both surfaces are C^2 and $\vec{F}$ is C^1,

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S_1} \vec{F} \cdot d\vec{s} = \int_{\partial S_2} \vec{F} \cdot d\vec{s} = \iint_{S_2} (\nabla \times \vec{F}) \cdot d\vec{S}\,.$$

Stokes' Theorem for Parametrized Surfaces

Theorem: Let S be an oriented surface defined by a 1-1 parametrization

$\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $\Phi(D) = S$, and D is a region to which Green's theorem applies. Let ∂S denote the oriented boundary of S and let $\vec{F}$ be a C^1 vector field on S. Then:

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$$

If S has no boundary (e.g. a sphere, ellipsoid, etc.) then the integral on the RHS is 0.

Ex. Evaluate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ where $\vec{F}(x, y, z) = <e^x, yz^2, sinz>$ and S is the unit sphere.

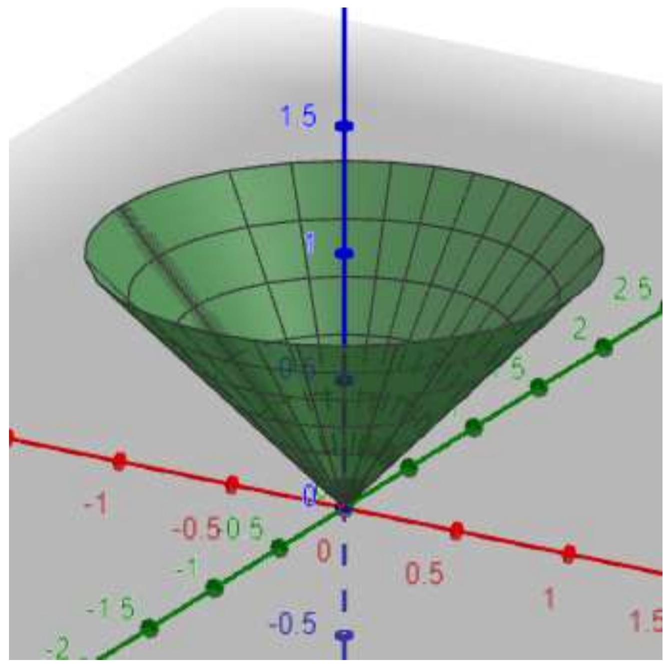
$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ but $\partial S = 0$ so

$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0.$

Ex. Verify Stokes' theorem for the portion of the cone given by

$\vec{\Phi}(r, \theta) = < rcos\theta,\ rsin\theta,\ r >; \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$ and the vector field

$\vec{F}(x, y, z) = < y, z, x >$.



We must show: $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$

RHS: First calculate $\nabla \times \vec{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\vec{\imath} - \vec{\jmath} - \vec{k}$$

$\vec{\Phi}(r, \theta) = < rcos\theta, rsin\theta, r >; \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$

$\vec{T}_r = < cos\theta, sin\theta, 1 > \qquad \vec{T}_\theta = < -rsin\theta, rcos\theta, 0 >$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ cos\theta & sin\theta & 1 \\ -rsin\theta & rcos\theta & 0 \end{vmatrix} = -rcos\theta\vec{\imath} - rsin\theta\vec{\jmath} + r\vec{k} .$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D (\nabla \times \vec{F}) \cdot (\vec{T}_r \times \vec{T}_\theta) dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 < -1, -1, -1 > \cdot < -r cos\theta, -r sin\theta, r > dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r cos\theta + r sin\theta - r)\, dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} r^2 cos\theta + \frac{1}{2} r^2 sin\theta - \frac{1}{2} r^2 \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} (\frac{1}{2} cos\theta + \frac{1}{2} sin\theta - \frac{1}{2})\, d\theta = -\pi.$$

LHS: $\vec{c}(t) =< cost, sint, 1 >; \quad 0 \leq t \leq 2\pi.$

$\vec{c}\,'(t) =< -sint, cost, 0 >; \qquad \vec{F}(\vec{c}(t)) =< sint, 1, cost >$

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} < sint, 1, cost > \cdot < -sint, cost, 0 > dt$$

$$= \int_0^{2\pi} (-\sin^2 t + cost) dt = \int_0^{2\pi} (-\frac{1}{2} + \frac{1}{2} cos2t + cost) dt = -\pi.$$

Notice that if we were trying to find $\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S}$ where S_1 were the surface given by $x^2 + y^2 \leq 1,\ z = 1,$ i.e., the disk of radius 1 in the plane $z = 1$, and $\vec{F}$ were the vector field in the previous example, we would already know the answer. This is because $\partial S = \partial S_1$ (they are both the circle of radius 1 in the plane $z = 1$). Thus by Stokes' theorem

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{\partial S_1} \vec{F} \cdot d\vec{s} = \iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S} = -\pi.$$