The General Lebesgue Integral

If f is an extended real valued function on E define:

$f^+(x) = \max\{f(x), 0\} \geq 0$

$f^-(x) = \max\{-f(x), 0\} \geq 0$

$f(x) = f^+(x) - f^-(x)$

Notice that f is measurable if and only if f^+ and f^- are measurable.

Prop. Let f be a measurable function on E. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.

Proof: Assume f^+ and f^- are integrable over E.

Notice that $|f| = f^+ + f^-$.

Thus $\int_E |f| = \int_E f^+ + \int_E f^- < \infty$,

and $|f|$ is integrable over E.

Now assume $|f|$ is integrable over E.

$0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$.

So by monotonicity: $\int_E f^+ \leq \int_E |f| < \infty$, $\int_E f^- \leq \int_E |f| < \infty$.

Thus f^+ and f^- are integrable over E.

Def. A measurable function f on E is said to be **integrable over $\boldsymbol{E}$** if $|f|$ is integrable over E. When this is so we define:

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Notice if f is nonnegative, i.e. $f = |f| = f^+,\ f^- = 0$ and we get the usual definition of the Lebesgue integral of a nonnegative function. If f is a bounded measurable function of finite support by linearity of integration this definition coincides with the original definition.

Notice also that, unlike Riemann integration, in order for a function f to be Lebesgue integrable we require $|f|$ to also be integrable.

Ex. $f(x) = \frac{sinx}{x}$ is integrable as a Riemann integral over $[0, \infty)$, but not as a Lebesgue integral because $\int_{[0,\infty)} |\frac{sinx}{x}| = \infty$.

Prop. Let f be integrable over E. Then f is finite a.e. on E and

$$\int_E f = \int_{E\sim A} f \text{ if } A \subseteq E \text{ and } m(A) = 0.$$

Proof: We know if g is nonnegative and g is integrable over E then g is finite a.e. on E. Thus $|f|$ is finite a.e. on E and hence f is.

Since f is integrable: $\int_E f = \int_E f^+ - \int_E f^-$

and $\int_E f^+ = \int_{E\sim E_0} f^+ \qquad \int_E f^- = \int_{E\sim E_0} f^-.$

So $\int_{E\sim E_0} f = \int_{E\sim E_0} f^+ - \int_{E\sim E_0} f^- = \int_E f^+ - \int_E f^- = \int_E f.$

Prop. (The integral comparison test). Let f be measurable on E. Suppose there is a nonnegative function g that is integrable over E and $|f| \leq g$ on E. Then f is integrable over E and

$$\left| \int_E f \right| \leq \int_E |f|.$$

Proof: By monotonicity of integrals for nonnegative measurable functions:

$$\int_E |f| \leq \int_E g < \infty,$$

so f is integrable.

Since $|f|$ is integrable, so are f^+ and f^-.

By the triangle inequality we have:

$$\left| \int_E f \right| = \left| \int_E f^+ - \int_E f^- \right| \leq \int_E f^+ + \int_E f^- = \int_E |f|.$$

Theorem: Let f and g be integrable over E. Then

1. for $\alpha, \beta \in \mathbb{R}$ $\quad \int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$
2. if $f \leq g$ on E then

$$\int_E f \leq \int_E g.$$

Corollary: Let f be integrable over E. Assume A and B are disjoint subsets of E.

Then: $$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof: $|(f)(\chi_A)| \leq |f|$ and $|(f)(\chi_B)| \leq |f|$ on E, thus $(f)(\chi_A)$ and $(f)(\chi_B)$ are integrable over E by the integrable comparison test.

Notice that: $(f)(\chi_{A\cup B}) = (f)(\chi_A) + (f)(\chi_B)$ on E.

But for any measurable subset D of E:

$$\int_D f = \int_D (f)(\chi_D).$$

Thus $$\int_{A\cup B} f = \int_{A\cup B} (f)(\chi_{A\cup B}) = \int_{A\cup B} \big((f)(\chi_A) + (f)(\chi_B)\big)$$

$$= \int_{A\cup B} (f)(\chi_A) + \int_{A\cup B} (f)(\chi_B) = \int_A f + \int_B f.$$

The Lebesgue Dominated Convergence Theorem: Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and $|f_n| \leq g$ on E for all n. If $f_n \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof: Since $|f_n| \leq g$ on E for all n then $|f| \leq g$ a.e. on E.

Since g is integrable over E, then f is integrable over E by the integral comparison test.

Since $\{f_n\}$ and f are integrable over E these functions are finite a.e. on E.

By removing sets where any of those functions are not finite (sets of measure 0), we can assume all of those functions are finite on E.

$g - f$ and $g - f_n$ are measurable nonnegative functions and $\{g - f_n\}$ converges to $g - f$ a.e. on E.

By Fatou's lemma:

$$\int_E (g - f) \le liminf \int_E (g - f_n).$$

Thus we can say:

$$\int_E g - \int_E f = \int_E (g - f) \le liminf \int_E (g - f_n)$$

$$= \int_E g - liminf \int_E f_n.$$

So we have:

$$-\int_E f \le -liminf \int_E f_n$$

Or:

$$\int_E f \ge limsup \int_E f_n. \qquad (*)$$

Notice that $g + f_n \ge 0$, so by Fatou's lemma:

$$\int_E (g + f) \le liminf \int_E (g + f_n)$$

$$\int_E g + \int_E f \le \int_E g + liminf \int_E f_n$$

$$\int_E f \le liminf \int_E f_n.$$

So together with $(*)$ we get $\lim_{n \to \infty} \int_E f_n = \int_E f$.

General Lebesgue Dominated Convergence Theorem: Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence of nonnegative measurable functions $\{g_n\}$ on E where $g_n \to g$ pointwise a.e. on E and $|f_n| \le g_n$ on E for all n.

If $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$ then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof: Just replace $\{g - f_n\}$ and $\{g + f_n\}$ with $\{g_n - f_n\}$ and $\{g_n + f_n\}$ in the proof of the Lebesgue Dominated Convergene Theorem.

Ex. Let f be a real valued integrable function on $[0,1]$. Show $x^n f(x)$ is integrable on $[0,1]$ and calculate $\lim_{n\to\infty} \int_0^1 x^n f(x)$.

Since $0 \le x \le 1$, $|x^n f(x)| \le |x^n||f(x)| \le |f(x)|$.

So by the integral comparison test since f is integrable over $[0,1]$ so is $x^n f(x)$ ($x^n f(x)$ is measurable because $f(x)$ and x^n are).

Since $f(x)$ is integrable over $[0,1]$, f is finite a.e. on $[0,1]$.

Thus we have: $\lim_{n\to\infty} x^n f(x) = 0$ a.e. on $[0,1]$.

By the Lebesgue dominated convergence theorem:

$$\lim_{n\to\infty} \int_0^1 x^n f(x) = \int_0^1 \lim_{n\to\infty} x^n f(x) = 0.$$

Ex. Evaluate $\lim_{n\to\infty} \int_E \frac{e^{-\frac{x}{n}}}{1+x^2}$ for $E = [0, \infty)$.

Let $f_n(x) = \frac{e^{-\frac{x}{n}}}{1+x^2}$ for $0 \leq x < \infty$.

$\lim_{n\to\infty} \frac{e^{-\frac{x}{n}}}{1+x^2} = \frac{1}{1+x^2}$; so $f_n(x) \to f(x) = \frac{1}{1+x^2}$ pointwise on $0 \leq x < \infty$.

Notice that $|f_n(x)| = \left|\frac{e^{-\frac{x}{n}}}{1+x^2}\right| \leq \frac{1}{1+x^2}$.

Let's let $g(x) = \frac{1}{1+x^2}$ and show that $g(x)$ is integrable over $E = [0, \infty)$.

Let $g_n(x) = \frac{1}{1+x^2}$ if $0 \leq x \leq n$

$= 0$ if $n < x$.

$\{g_n(x)\}$ is increasing, measurable, and $g_n(x) \to g(x)$ pointwise on E.

Notice that each g_n is Riemann integrable over $[0, n]$, so the Lebesgue integral equals the Riemann integral over $[0, n]$.

Thus $\int_E g_n = \int_0^n \frac{1}{1+x^2} dx = tan^{-1}(n)$.

Now by the Lebesgue monotone convergence theorem:

$$\lim_{n\to\infty} \int_E g_n = \int_E \frac{1}{1+x^2}$$

$$\lim_{n\to\infty} tan^{-1}(n) = \int_E \frac{1}{1+x^2}$$

$$\frac{\pi}{2} = \int_E \frac{1}{1+x^2}.$$

So $g(x)$ is integrable over E.

Since $\{f_n\}$ are all measurable, we can apply the Lebesgue dominated convergence theorem to get:

$$\lim_{n\to\infty} \int_E \frac{e^{-\frac{x}{n}}}{1+x^2} = \int_E \frac{1}{1+x^2} = \frac{\pi}{2}.$$