Closed and Exact Differential Forms

Def. A differential k-form ω is called **closed** if $d\omega = 0$.

Ex. Let $\omega = (x^2 + y^2)dx + 2xydy$. Show that ω is closed.

$$\begin{aligned} d\omega &= d[(x^2 + y^2)dx + 2xydy] \\ &= d[(x^2 + y^2)dx] + d[2xydy] \\ &= d(x^2 + y^2) \wedge dx + d(2xy) \wedge dy \\ &= \left(\frac{\partial}{\partial x}(x^2 + y^2)dx + \frac{\partial}{\partial y}(x^2 + y^2)dy\right) \wedge dx \\ &\qquad + \left(\frac{\partial}{\partial x}(2xy)dx + \frac{\partial}{\partial y}(2xy)dy\right) \wedge dy \\ &= (2xdx + 2ydy) \wedge dx + (2ydx + 2xdy) \wedge dy \\ &= 2ydy \wedge dx + 2ydx \wedge dy = 0. \end{aligned}$$

Ex. Show that any 2 form on $\mathbb{R}^2$ is closed.

Any 2 form on $\mathbb{R}^2$, ω, can be written as $\omega = f(x,y)dx \wedge dy$.

$$\begin{aligned} d\omega &= d(f(x,y)dx \wedge dy) \\ &= df \wedge dx \wedge dy \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx \wedge dy \\ &= \frac{\partial f}{\partial x}dx \wedge dx \wedge dy + \frac{\partial f}{\partial y}dy \wedge dx \wedge dy = 0. \end{aligned}$$

Ex. Show that $\omega = dx_i \wedge dx_j$ is closed as a 2 form on $\mathbb{R}^n$.

$$d\omega = d\left(dx_i \wedge dx_j\right) = d(dx_i) \wedge dx_j + (-1)^1 dx_i \wedge d\left(dx_j\right) = 0.$$

By induction one can show that $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$ is closed on $\mathbb{R}^n$.

Def. A differential k-form ω is called **exact** if $\omega = d\eta$ for some $(k-1)$-form η.

Ex. Show that $\omega = (x^2 + y^2)dx + 2xydy$ is exact on $\mathbb{R}^2$.

So we have to show we can find a real valued function f on $\mathbb{R}^2$ such that $df = \omega = (x^2 + y^2)dx + 2xydy$.

However, we know that:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

So we have to find a function f such that:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (x^2 + y^2)dx + 2xydy\ .$$

Thus we need to have:

$$\frac{\partial f}{\partial x} = x^2 + y^2$$
$$\frac{\partial f}{\partial y} = 2xy.$$

We solve these 2 equations as was done in second year calculus.

$$f(x, y) = \int (x^2 + y^2)dx = \frac{x^3}{3} + xy^2 + g(y).$$

Now differentiate this equation with respect to y.

$$\frac{\partial f}{\partial y} = 2xy + g'(y).$$

But we also know that $\frac{\partial f}{\partial y} = 2xy$, so

$$2xy + g'(y) = 2xy.$$

Thus $g'(y) = 0$ and $g(y) = c$.

Thus if $f(x, y) = \frac{x^3}{3} + xy^2 + c$, then $df = \omega = (x^2 + y^2)dx + 2xydy$.

Notice that if ω is exact (i.e. $\omega = d\eta$), then it must be closed since:

$$d\omega = d(d\eta) = 0$$

So exact ⇒ closed. However, if ω is closed does that imply it's exact? This is actually a very deep question. The answer depends on the set that ω is defined on.

Ex. Suppose $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, is a 1-form defined on $\mathbb{R}^2 - (0,0)$. Show ω is closed.

$$d\omega = d\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$$

$$= -d\left(\frac{y}{x^2+y^2}dx\right) + d\left(\frac{x}{x^2+y^2}dy\right)$$

$$= -d\left(\frac{y}{x^2+y^2}\right) \wedge dx + d\left(\frac{x}{x^2+y^2}\right) \wedge dy$$

$$= -\left[\frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}\ dy \wedge dx\right] + \left[\frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2}\ dx \wedge dy\right]$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}\ dx \wedge dy + \frac{y^2-x^2}{(x^2+y^2)^2}\ dx \wedge dy = 0.$$

Is this ω exact? That is, is there a smooth function (or C^1) such that $df = \omega$?

Suppose there is a smooth function, f, on $\mathbb{R}^2 - (0,0)$ such that $\omega = df$

We can transform $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ into polar coordinates by:

$$g: \mathbb{R}^2 \to \mathbb{R}^2$$
$$(r,\theta) \to (r\cos\theta, r\sin\theta)$$

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$

Now let's calculate:

$$g^*\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$$
$$= \frac{-y}{x^2+y^2} \circ g\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) + \frac{x}{x^2+y^2} \circ g\left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$$
$$= \frac{-r\sin\theta}{r^2}(\cos\theta\, dr - r\sin\theta\, d\theta) + \frac{r\cos\theta}{r^2}(\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= d\theta$$

So it looks like $\omega = d\theta$, but θ is not continuous on $\mathbb{R}^2 - (0,0)$, as:

$$\lim_{\theta \to 2\pi} \theta = 2\pi \neq 0$$

Furthermore, if there was a smooth function, f, on $\mathbb{R}^2 - (0,0)$ such that $df = \omega$, then:

$$df = d\theta$$
$$d(f - \theta) = 0 \;\; \Rightarrow \;\; f = \theta + \text{constant}$$

Hence f can't be continuous on $\mathbb{R}^2 - (0,0)$ because θ isn't. Thus, there is no smooth (or C^1) function, f, on $\mathbb{R}^2 - (0,0)$ with $df = \omega$. So ω is closed but not exact.

However, on some subsets of $\mathbb{R}^n$, $d\omega = 0$ does imply $\omega = d\eta$, for any closed k-form ω.

Theorem (Poincare's Lemma): If $A \subseteq \mathbb{R}^n$ is an open convex region, then every closed form on A is exact.

One way to prove this is to observe that if $\omega = \sum_{i=1}^{n} \omega_i \, dx_i$ is a 1-form and $\omega = df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$ (and we assume $f(0) = 0$), then we have:

$$f(x) = \int_0^1 \frac{d}{dt} f(tx)dt = f(x) - f(0).$$

If $u = tx$, then by the chain rule:

$$= \int_0^1 \sum_{i=1}^{n} \left(\frac{\partial}{\partial u_i} f(tx) \right) (x_i) \, dt$$

$$= \int_0^1 \sum_{i=1}^{n} \big(\omega_i(tx)\big) \, x_i \, dt.$$

So in order to find f given ω, we should look at:

$$\mathrm{I}\omega(x) = \int_0^1 \sum_{i=1}^{n} \big(\omega_i(tx)\big) \, x_i \, dt.$$

For a k-form (instead of a 1-form) we get:

$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} \; dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ and

$$\mathrm{I}\omega(x) = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{k} (-1)^{\alpha-1} \left(\int_0^1 t^{k-1} \, \omega_{i_1, \ldots, i_k}(tx) dt \right) x_{i_\alpha} dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_\alpha}} \wedge \ldots \wedge dx_{i_k}$$

where $\widehat{dx_{i_\alpha}}$ means omit dx_{i_α}.

Notice that I takes a k-form and gives us a k-1 form. It also has the property that $\mathrm{I}(0) = 0$. Through a very messy calculation one can show that:

$$\omega = \mathrm{I}(d\omega) + d(\mathrm{I}\omega)$$

Thus, if $d\omega = 0$, since $\mathrm{I}(0) = 0$ we have: $\omega = d(\mathrm{I}\omega)$ and ω is exact.

Let $A \subseteq \mathbb{R}^n$ be an open set. Let $\Omega^k(A)$ be the vector space of k-forms on A. We can create a sequence of linear maps between vector spaces by:

$$\Omega^0(A) \xrightarrow{d} \Omega^1(A) \xrightarrow{d} \Omega^2(A) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(A).$$

If $\omega \in \Omega^k(A)$, then ω is closed if ω is in the kernel of:

$$d\colon \Omega^k(A) \to \Omega^{k+1}(A)$$

and ω is exact if it's in the image of:

$$d\colon \Omega^{k-1}(A) \to \Omega^k(A).$$

Since $d^2(\eta) = 0$ for any η, the image of $d\colon \Omega^{k-1}(A) \to \Omega^k(A)$ is contained in the kernel of $d\colon \Omega^k(A) \to \Omega^{k+1}(A)$.

We can create a group, called the k^{th} de Rham cohomology group, $H^k_{dR}(A)$, by:

$$H^k_{dR}(A) = \frac{\ker(d\colon \Omega^k(A) \to \Omega^{k+1}(A))}{\mathrm{I}m\big(d\colon \Omega^{k-1}(A) \to \Omega^k(A)\big)}.$$

So an element of $H^k_{dR}(A)$ is a closed k-form on A. Two elements $(\alpha_1, \alpha_2 \in H^k_{dR}(A))$ are considered the same (i.e. they are in the same equivalence class) if they differ by an exact k-form:

$$\alpha_1 = \alpha_2 + d\eta \quad ; \quad \eta \text{ a } k\text{-1 form}$$

These groups are topological invariants. Thus, if A_1 is homeomorphic to A_2, then their de Rham cohomology groups will be the same.