

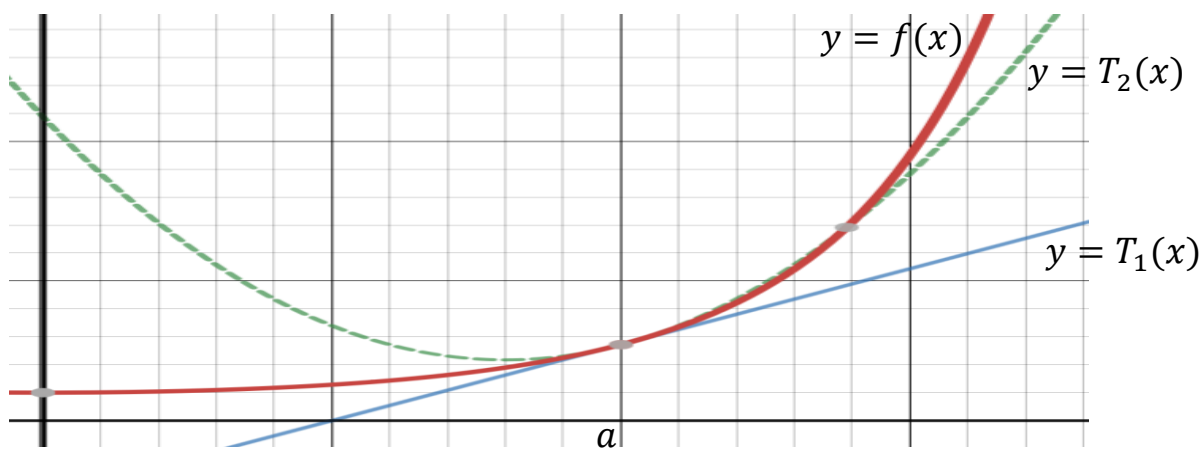
## Taylor Series in 2 Variables

Recall from first year calculus that if  $f(x)$  has an infinite number of derivatives near a point  $x = a$ , then we have:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^n(a)}{n!}(x - a)^n + R_n(x, a)$$

where  $R_n(x, a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$  and  $c$  is between  $x$  and  $a$ .

This allows us to approximate the value of a function,  $f(x)$ , using an  $n$ -th degree polynomial if we know the value of the function and its derivatives at  $x = a$ . In addition,  $R_n(x, a)$  = error in the approximation, allows us to put an upper bound on the error of this approximation.



where:

$$T_1(x) = f(a) + f'(a)(x - a)$$

$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2.$$

Ex. Approximate the value of  $e^{.02}$  using a second order Taylor polynomial around  $a = 0$ .

$$\begin{array}{ll} f(x) = e^x & f(0) = 1 \\ f'(x) = e^x & f'(0) = 1 \\ f''(x) = e^x & f''(0) = 1 \\ f'''(x) = e^x & f'''(0) = 1 \end{array}$$

$$e^x = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x^2) + R_2(x, 0)$$

where  $R_2(x, 0) = \frac{f'''(c)}{3!}(x - 0)^3 = \frac{e^c}{3!}x^3$  and  $0 < c < x$ .

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{e^c}{3!}x^3$$

So now we can plug in  $x = .02$

$$e^{.02} \approx 1 + .02 + \frac{(.02)^2}{2!} = 1 + .02 + \frac{.0004}{2} = 1.0202$$

The error is no bigger than:

$$|R_2(.02, 0)| = \frac{e^c}{3!}(.02)^3 \leq \frac{3}{3!}(.000008) = .000004.$$

Now we want to be able to approximate the value of  $f(x, y)$  using polynomials in  $x$  and  $y$ .

Second Order Taylor Formula:

Let  $f: U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  have continuous partial derivatives of third order and let  $(x_0, y_0) \in U$ . Then we have:

$$f(x, y) = f(x_0, y_0) + (f_x(x_0, y_0))(x - x_0) + (f_y(x_0, y_0))(y - y_0) \\ + \frac{1}{2} [f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \\ f_{yy}(x_0, y_0)(y - y_0)^2] + R_2(x, y, x_0, y_0)$$

where:

$$R_2(x, y, x_0, y_0) = \frac{1}{3} \sum_{i,j,k=1}^2 \left( \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} (c_{ijk}) \right) (\Delta x_i) (\Delta x_j) (\Delta x_k)$$

$$x_1 = x$$

$$x_2 = y$$

$$\Delta x_i = x - x_0 \text{ if } i = 1 \\ = y - y_0 \text{ if } i = 2$$

and  $c_{ijk}$  lies somewhere on the line segment between  $(x, y)$  and  $(x_0, y_0)$ .

Ex. Compute the second order Taylor formula for  $f(x, y) = e^x \cos y$  about the point  $(x_0, y_0) = (0, 0)$  and approximate the value of  $e^{.02} \cos(.04)$ .

$$\begin{array}{ll}
 f(x, y) = e^x \cos y & f(0, 0) = 1 \\
 f_x = e^x \cos y & f_x(0, 0) = 1 \\
 f_y = -e^x \sin y & f_y(0, 0) = 0 \\
 \\ 
 f_{xx} = e^x \cos y & f_{xx}(0, 0) = 1 \\
 f_{xy} = -e^x \sin y & f_{xy}(0, 0) = 0 \\
 f_{yy} = -e^x \cos y & f_{yy}(0, 0) = -1
 \end{array}$$

$$\begin{aligned}
 e^x \cos y &= 1 + 1(x - 0) + 0(y - 0) \\
 &+ \frac{1}{2}[1(x - 0)^2 + 2(0)(x - 0)(y - 0) - 1(y - 0)^2] + R_2(x, y, 0, 0) \\
 &= 1 + x + x^2 + R_2(x, y, 0, 0).
 \end{aligned}$$

Thus we have:

$$e^x \cos y \approx 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$$

$$\begin{aligned}
 e^{.02} \cos(.04) &\approx 1 + .02 + \frac{1}{2}(.02)^2 - \frac{1}{2}(.04)^2 \\
 &= 1 + .02 + .0002 - .0008 = 1.0194.
 \end{aligned}$$

Ex. Determine the second order Taylor formula for  $f(x, y) = x \cos y$  about the point  $(1, \frac{\pi}{4})$  and approximate:

$$f\left(1.1, \frac{\pi}{4} + .2\right) = (1.1) \cos\left(\frac{\pi}{4} + .2\right).$$

$$\begin{array}{ll} f(x, y) = x \cos y & f\left(1, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\ f_x = \cos y & f_x\left(1, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\ f_y = -x \sin y & f_y\left(1, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\ f_{xx} = 0 & f_{xx}\left(1, \frac{\pi}{4}\right) = 0 \\ f_{xy} = -\sin y & f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\ f_{yy} = -x \cos y & f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \end{array}$$

$$\begin{aligned} f(x, y) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - 1) - \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) \\ &+ \frac{1}{2}\left[0(x - 1)^2 + 2\left(-\frac{\sqrt{2}}{2}\right)(x - 1)\left(y - \frac{\pi}{4}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(y - \frac{\pi}{4}\right)^2\right] \\ &\quad + R_2\left(x, y, 1, \frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - 1) - \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) \\ &\quad + \frac{1}{2}\left[-\sqrt{2}(x - 1)\left(y - \frac{\pi}{4}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(y - \frac{\pi}{4}\right)^2\right] + R_2\left(x, y, 1, \frac{\pi}{4}\right). \end{aligned}$$

$$\begin{aligned} (1.1) \cos\left(\frac{\pi}{4} + .2\right) &\approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(.1) - \frac{\sqrt{2}}{2}(.2) + \frac{1}{2}\left[-\sqrt{2}(.1)(.2) + \left(-\frac{\sqrt{2}}{2}\right)(.2)^2\right] \\ &= (.9) \frac{\sqrt{2}}{2} - (.04) \left(\frac{\sqrt{2}}{2}\right) \approx .6081. \end{aligned}$$