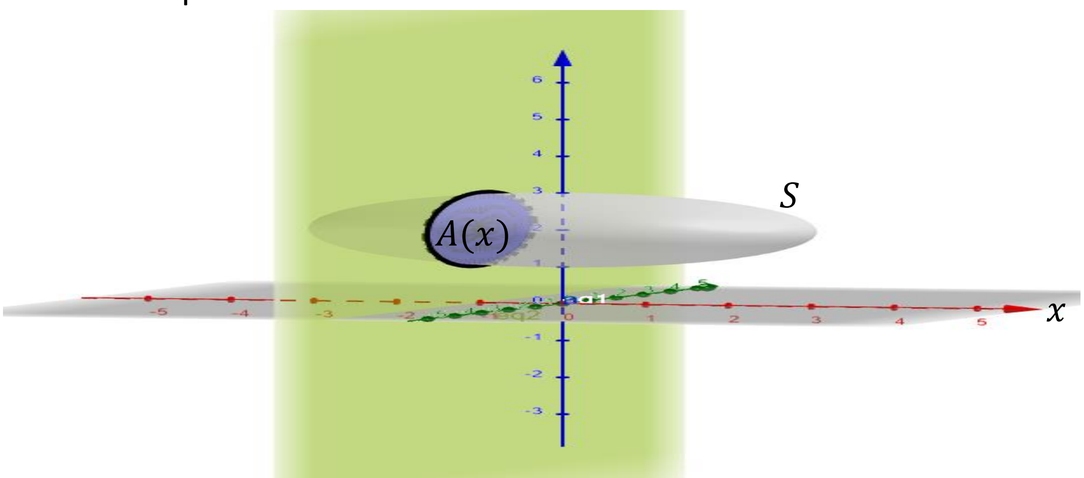
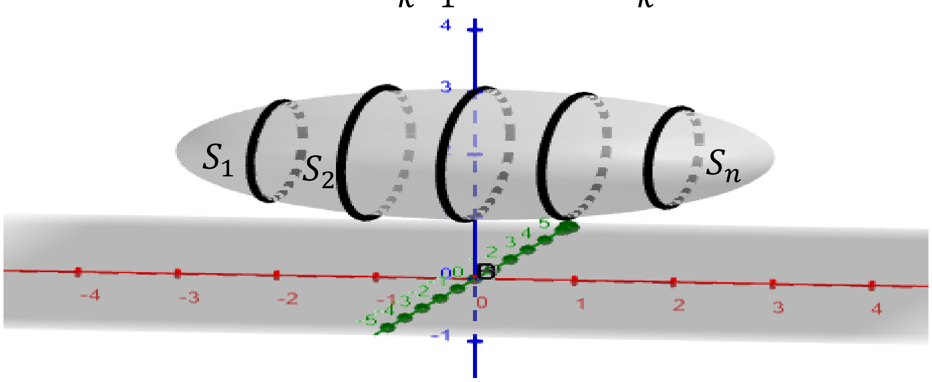
Volumes: Integrating Cross-sections

Suppose we start with a solid S in 3-space and slice it with a plane perpendicular to the x-axis. The intersection of the plane and the solid, called a cross-section of S, will generally have different areas, $A(x)$, depending on which point, x, along the x-axis the plane intersects.



Now let's partition the x-axis into $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where the solid, S, lies between the plane that intersects the x-axis at $x = a$ and $x = b$. Call ΔV_k the volume of the solid S that lies between the planes that intersect the x-axis at $x = x_{k-1}$ and $x = x_k$.



$\Delta V_k \approx A(x_k^*)\Delta x_k$, where $\Delta x_k = x_k - x_{k-1}$, and x_k^* is any point between x_{k-1} and x_k. Meaning ΔV_k is approximately equal to the area of the base, $A(x_k^*)$, times the height, $\Delta x_k = x_k - x_{k-1}$. Now to get the volume of S we add up all of the $\Delta V_k{}'$s and take a limit as $\Delta x_k \to 0$.

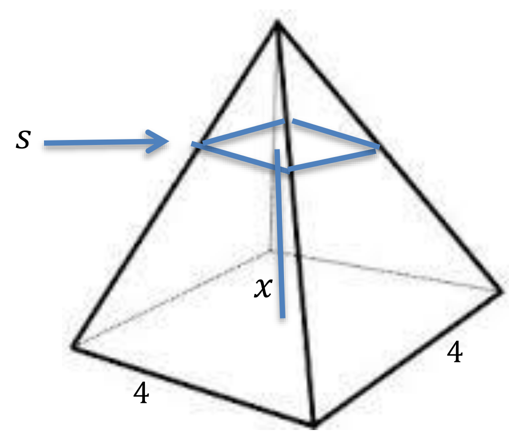
Def. Formula for the volume S given $A(x)$ is the cross-sectional area of S:

$$\boldsymbol{V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_{i-1})\Delta x_i = \int_{x=a}^{x=b} A(x)\, dx\,.}$$

Notice that there is nothing special about cutting the solid with planes parallel to the x-axis. If $A(y)$ is the cross-sectional area of a solid S when S is intersected with a plane perpendicular to the y-axis and the solid lines between the planes that intersect the y-axis at $y = c$ and $y = d$, then the volume is given by:

$$V = \int_{y=c}^{y=d} A(y)\, dy\,.$$

Ex. Find the volume of a pyramid with a square base if the base is $4m \times 4m$ and the height is $6m$.



By similar triangles $\frac{s}{4} = \frac{6-x}{6}$ or $s = \frac{2}{3}(6-x)$. Thus $A(x) = s^2 = \frac{4}{9}(6-x)^2$.

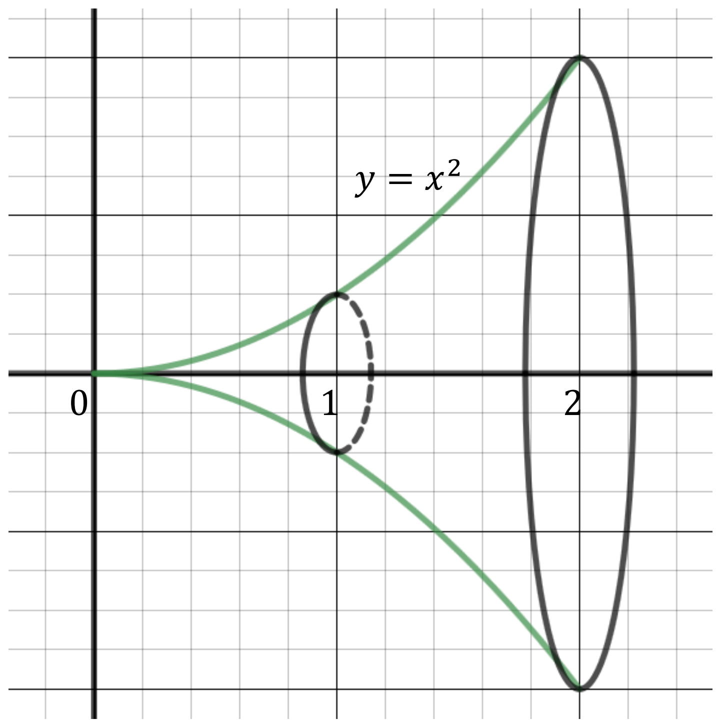
$$V = \int_{x=0}^{x=6} \frac{4}{9}(6-x)^2\, dx = -\frac{4}{9}\left(\frac{(6-x)^3}{3}\right)\Bigg|_{x=0}^{x=6}$$

$$= -\frac{4}{9}\left[0 - \frac{6^3}{3}\right] = 32m^3\,.$$

Integrating the cross-sectional area of a solid can be a very useful method for finding the volume of a solid of revolution, meaning a solid generated by taking a two dimensional region and revolving it about a line.

Ex. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$, $y = 0$, and $x = 2$ about the x-axis.

Notice that the cross-sections are all disks whose radius is $y = x^2$.

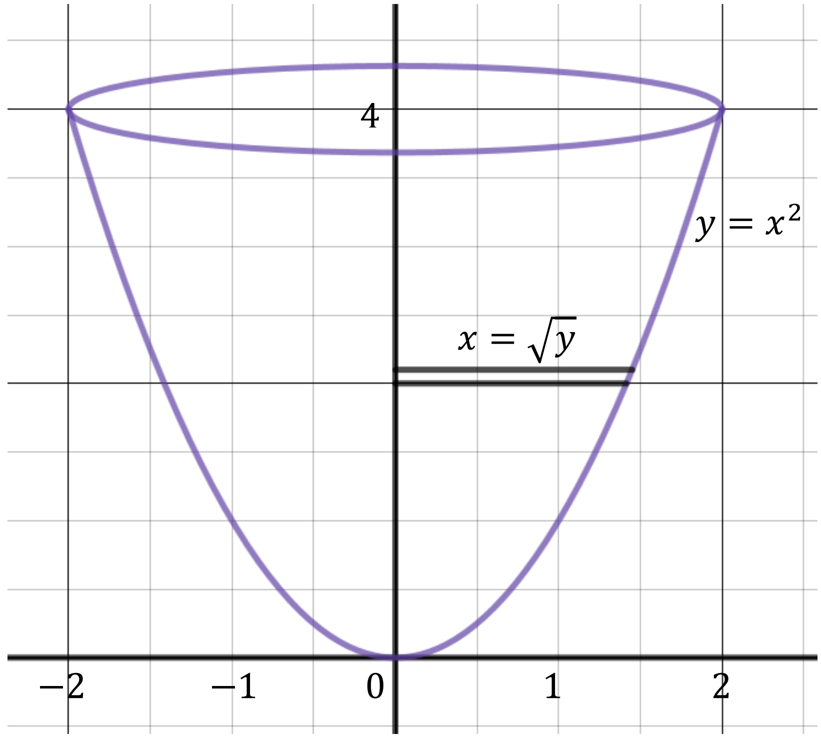


Thus, $A(x) = \pi(y)^2 = \pi(x^2)^2$.

$$V = \int_0^2 \pi(x^2)^2\,dx = \pi\int_0^2 x^4\,dx = \pi\left(\frac{x^5}{5}\right)\Bigg|_0^2 = \frac{32\pi}{5}.$$

This method of finding the volume of a solid of revolution is often called the "disk" method because all of the cross-sections are disks.

Ex. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = x^2, x = 0$, and $y = 4$.



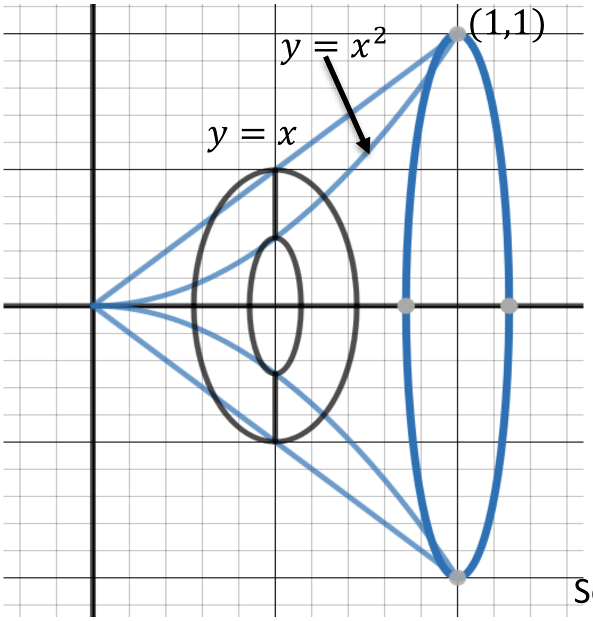
Notice that when we cut this solid with planes perpendicular to the y-axis we once again get disks.

The radius of the disk for a given y is the positive x value that corresponds to it on the curve $y = x^2$, that is $x = \sqrt{y}$.

Thus, $A(y) = \pi x^2 = \pi\left(\sqrt{y}\right)^2 = \pi y$.

$$V = \int_{y=0}^{y=4} \pi y \, dy = \pi \frac{y^2}{2}\Big|_{y=0}^{y=4} = 8\pi \, .$$

Ex. The region between $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

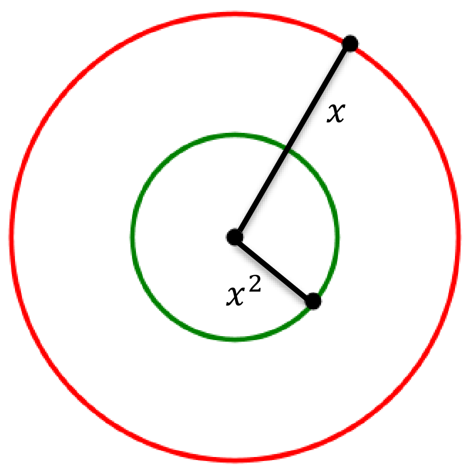


First, let's find where the curves $y = x$ and $y = x^2$ intersect.

$$x = x^2 \Rightarrow x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1.$$

So the curves intersect at $(0,0)$ and $(1,1)$.

When we slice this solid with a plane perpendicular to the x-axis we don't get a disk. However, we do get an annulus where the inner radius is the distance from the x-axis to the "bottom" curve, $y = x^2$, and the outer radius is the distance from the x-axis to the "top" curve, $y = x$.



The area of an annulus is given by:

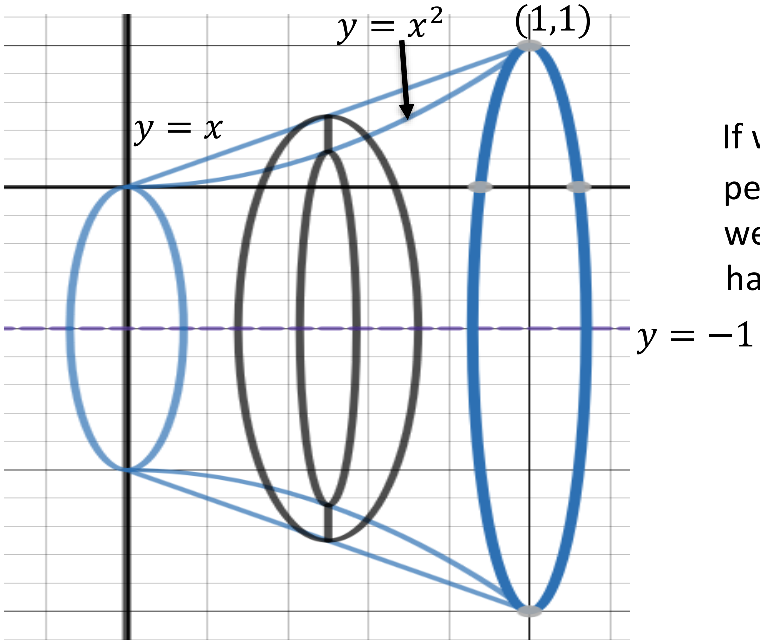
$$A = \pi r_2^2 - \pi r_1^2 = \pi x^2 - \pi (x^2)^2 = \pi x^2 - \pi x^4 \,.$$

$$V = \int_{x=0}^{x=1} (\pi x^2 - \pi x^4)\, dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Bigg|_{x=0}^{x=1} = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$$

Since the annulus with a little thickness to it looks like a washer, this method is often called the "washer" method.

If you take a region in the x- y plane and rotate it about a line parallel to the x-axis (i.e. $y = c$) or a line parallel to the y-axis (i.e. $x = a$) and slice it with a **plane perpendicular to the line of rotation**, then you will get cross-sections that are disks or annuli.

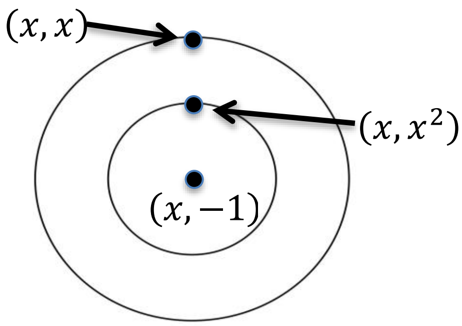
Ex. Find the volume of the solid object obtained by rotating the region between $y = x$ and $y = x^2$ about the line $y = -1$.

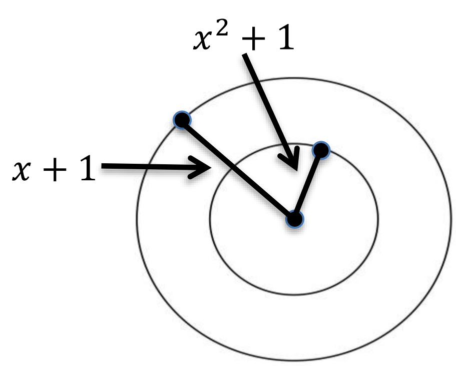


If we slice this solid with a plane perpendicular to the line $y = -1$, we will get annuli. However, we have to be careful when we calculate the inner and outer radii of the annulus.

If we fix the x-coordinate at x, then the points on the two curves are (x, x^2) and (x, x). The radii are gotten by taking the absolute value of the difference in the y-coordinates of the curves.

$r_1 = x^2 - (-1) = x^2 + 1$ $\qquad$ $r_2 = x - (-1) = x + 1$





So the cross-sectional area is:

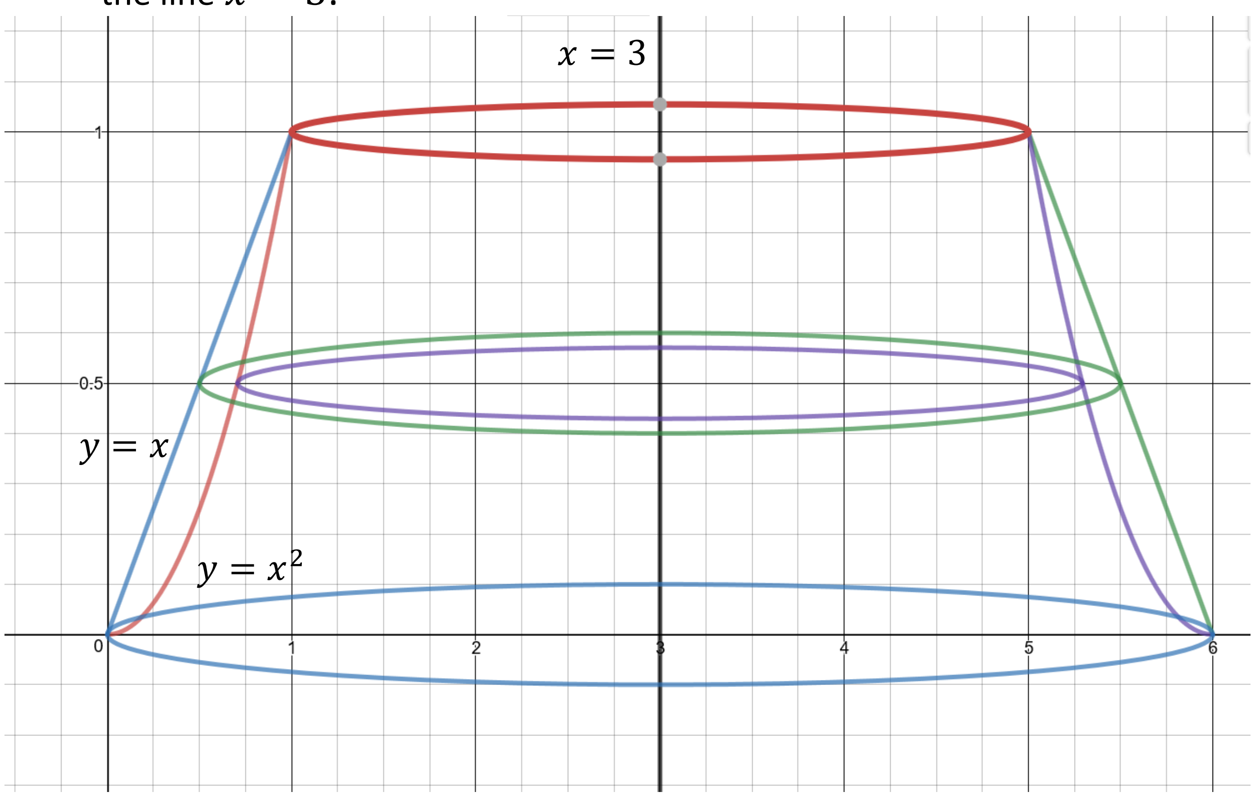
$$A(x) = \pi(r_2^2 - r_1^2) = \pi((x+1)^2 - (x^2+1)^2)$$

$$V = \pi \int_{x=0}^{x=1} ((x+1)^2 - (x^2+1)^2)\, dx = \pi \int_0^1 (-x^4 - x^2 + 2x)\, dx$$

$$= \pi \left(-\frac{x^5}{5} - \frac{x^3}{3} + x^2 \right) \Bigg|_{x=0}^{x=1} = \pi \left(-\frac{1}{5} - \frac{1}{3} + 1 \right)$$

$$= \pi \left(-\frac{8}{15} + 1 \right) = \frac{7\pi}{15}.$$

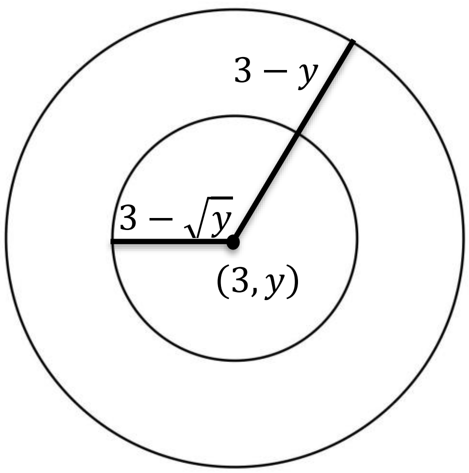
Ex. Find the volume from rotating the region between $y = x$ and $y = x^2$ about the line $x = 3$.



Now we slice the solid with planes perpendicular to the line $x = 3$.
The planes will go from $y = 0$ to $y = 1$.

Thus, the integral will be in terms of y. For a fixed y, the coordinates on the two curves are (y, y) and $(\sqrt{y}, y)$.

Thus, the annulus looks like:



$r_2 = 3 - y$

$r_1 = 3 - \sqrt{y}$

$$A(y) = \pi(r_2^2 - r_1^2) = \pi\left((3 - y)^2 - \left(3 - \sqrt{y}\right)^2\right)$$

$$V = \pi \int_{y=0}^{y=1} \left((3 - y)^2 - \left(3 - \sqrt{y}\right)^2\right) dy$$

$$= \int_{y=0}^{y=1} [9 - 6y + y^2 - \left(9 - 6\sqrt{y} + y\right)]\, dx$$

$$= \int_{y=0}^{y=1} -7y + 6\sqrt{y} + y^2\, dy$$

$$= -\frac{7}{2}y^2 + 6\left(\frac{2}{3}\right)y^{\frac{3}{2}} + \frac{y^3}{3}\Bigg|_0^1$$

$$= -\frac{7}{2} + 4 + \frac{1}{3}$$

$$= \frac{5}{6}\,.$$