The Lebesgue Integral $\int_E f$: $f \geq 0$ - HW Problems

1. Find $\int_{[0,8]} \frac{1}{\sqrt[3]{x}}$ as a Lebesgue integral. Do this by finding a sequence $\{f_n\}$ of increasing bounded measurable functions on [0,8] that converge pointwise to $\frac{1}{\sqrt[3]{x}}$. Since f_n is bounded for all n, and the interval is finite, the Lebesgue value for $\int_{[0,8]} f_n$ equals the Riemann integral value. Justify why you can take the limit.

2. Suppose $m(E) = 0$ and $f \equiv \infty$ on E. Show that $\int_E f = 0$.

3. Suppose that $f \geq 0$ and integrable over E. Show that given any $\epsilon > 0$ there is a simple function φ on E with finite support and such that $\int_E |f - \varphi| < \epsilon$. Furthermore, show that if E is a closed, bounded interval that there is a step function s on E with finite support such that $\int_E |f - s.| < \epsilon$.

4. Suppose that $f \geq 0$ and measurable over E.

a. Show that there is an increasing sequence φ_n of nonnegative simple functions with finite support on E which converges pointwise to f on E.

b. Show that $\int_E f =$
$\sup\{\int_E \varphi \mid \varphi \text{ is simple with finite support, and } 0 \leq \varphi \leq f \text{ on } E\}$.

5. Suppose that $\{g_n\}$ is a sequence of nonnegative measurable functions on E that converges pointwise to g on E. In addition, suppose that $g_n \leq g$ on E for each n. Prove that $\lim_{n\to\infty} \int_E g_n = \int_E g$.

6. Find an example that shows that the monotone convergence theorem is not true for a decreasing sequence of functions.