The First Fundamental Form: Lengths of Curves on Surfaces

When studying a surface, S, we would like to be able to say how to find the length of a curve on S and how to find the surface area of S. The first fundamental form will allow us to do that.

We already know that if γ is a curve on S, i.e., $\gamma: (a, b) \to S \subseteq \mathbb{R}^3$ then we can find its length by:

$$L = \int_a^b \|\gamma'(t)\| \, dt.$$

Def. Let $p \in S$, a point on a surface S. **The first fundamental form of S at p** associates to the tangent vectors $\vec{w}_1, \vec{w}_2 \in T_pS$ the scalar product:

$$< \vec{w}_1, \vec{w}_2 >_{p,S} = \vec{w}_1 \cdot \vec{w}_2.$$

Thus, the first fundamental form is just the usual dot product in $\mathbb{R}^3$ restricted to vectors in T_pS.

Let $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S$ be a surface patch for S and

$\vec{\Phi}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$.

We know $\vec{\Phi}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ and $\vec{\Phi}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$ are tangent vectors in T_pS and they are linearly independent ($\vec{\Phi}_u \times \vec{\Phi}_v \neq \vec{0}$) and hence any vector $\vec{w} \in T_pS$ can be written as:

$$\vec{w} = a\vec{\Phi}_u + b\vec{\Phi}_v \, ; \;\; a, b \in \mathbb{R}.$$

We define maps:

$du: T_pS \to \mathbb{R}$ and $dv: T_pS \to \mathbb{R}$ by

$du(\vec{w}) = a$ and $dv(\vec{w}) = b$ where

$$\vec{w} = a\vec{\Phi}_u + b\vec{\Phi}_v \,.$$

$$< \vec{w}, \vec{w} > = a^2\left(\vec{\Phi}_u \cdot \vec{\Phi}_u\right) + 2ab\vec{\Phi}_u \cdot \vec{\Phi}_v + b^2(\vec{\Phi}_v \cdot \vec{\Phi}_v).$$

If we let:

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = \vec{\Phi}_v \cdot \vec{\Phi}_u$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v$$

then:

$$< \vec{w}, \vec{w} > = Ea^2 + 2Fab + Gb^2 \,.$$

But, $du(\vec{w}) = a$ and $dv(\vec{w}) = b$, so we can write:

$$< \vec{w}, \vec{w} > = E\left(du(\vec{w})\right)^2 + 2F\left(du(\vec{w})dv(\vec{w})\right) + G\left(dv(\vec{w})\right)^2.$$

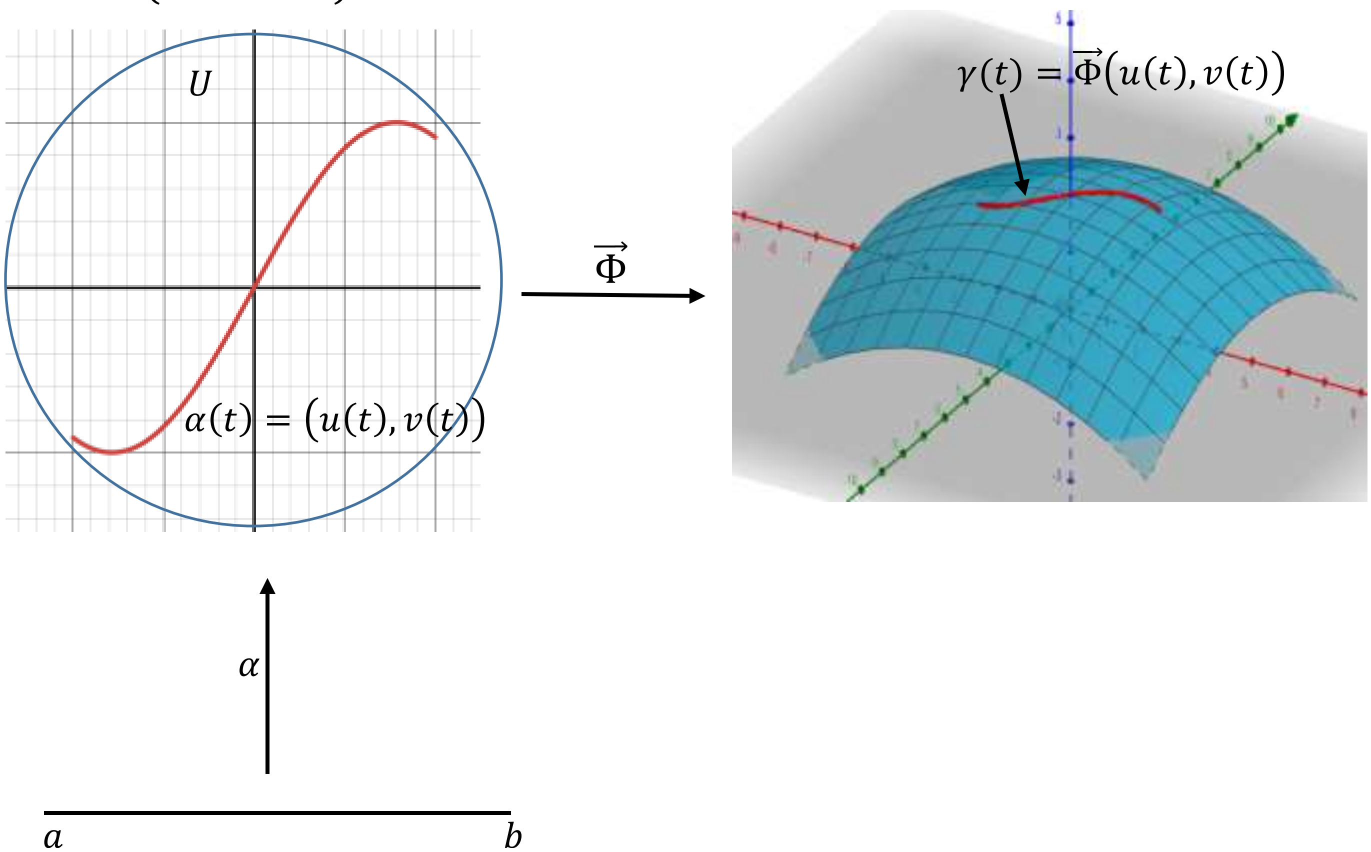
We call $Edu^2 + 2Fdudv + Gdv^2$ the first fundamental form of the surface patch $\vec{\Phi}(u, v)$.

$E, F, G, du,$ and dv all depend on the choice of $\vec{\Phi}$, but the first fundamental form itself only depends on S and $p \in S$.

Let γ be a curve on the surface, S, and $\vec{\Phi}(u,v)$ a surface patch. Then:

$$\gamma(t) = \vec{\Phi}\big(u(t), v(t)\big).$$

That is, we can think of γ as the image of a curve, α, in $U \subseteq \mathbb{R}^2$ under $\vec{\Phi}$, where $\alpha(t) = \big(u(t), v(t)\big)$.



By the chain rule:

$$\gamma'(t) = \vec{\Phi}_u u'(t) + \vec{\Phi}_v v'(t)$$

$$\gamma'(t) = u'(t)\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) + v'(t)\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

So we have:

$$< \gamma'(t), \gamma'(t) > = \|\gamma'(t)\|^2$$

$$= (u')^2 \vec{\Phi}_u \cdot \vec{\Phi}_u + 2u'v' \vec{\Phi}_u \cdot \vec{\Phi}_v + (v')^2 \vec{\Phi}_v \cdot \vec{\Phi}_v$$

$$= E(u')^2 + 2F(u')(v') + G(v')^2.$$

Thus we have:

$$L = \int_a^b \|\gamma'(t)\| \, dt = \int_a^b (E(u')^2 + 2F(u')(v') + G(v')^2)^{\frac{1}{2}} \, dt.$$

Because of the equation above, you sometimes see:

$$(*) \qquad ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

This means if $\gamma(t) = \vec{\Phi}\big(u(t), v(t)\big)$ is a curve on S and $s = s(t)$ is its arc length, then since $\frac{ds}{dt} = s' = \|\gamma'(t)\|$:

$$(s')^2 = E(u')^2 + 2F(u')(v') + G(v')^2.$$

We can also write $(*)$ in matrix form:

$$ds^2 = (du \quad dv) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = Edu^2 + 2Fdudv + Gdv^2.$$

If we let:

$$g_{11} = E = \vec{\Phi}_u \cdot \vec{\Phi}_u$$
$$g_{12} = F = \vec{\Phi}_u \cdot \vec{\Phi}_v$$
$$g_{21} = F = \vec{\Phi}_v \cdot \vec{\Phi}_u$$
$$g_{22} = G = \vec{\Phi}_v \cdot \vec{\Phi}_v$$

Then: $$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

is called the **metric tensor** for the parametrization $\vec{\Phi}$ of the surface, S.

Ex. Find the first fundamental form for the plane $z = 4x + 3y + 5$, given by the parametrization:

$$\vec{\Phi}(u, v) = (u, v, 4u + 3v + 5), \quad (u, v) \in \mathbb{R}^2.$$

$$\vec{\Phi}_u = (1, 0, 4), \qquad \vec{\Phi}_v = (0, 1, 3)$$
$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = (1, 0, 4) \cdot (1, 0, 4) = 17$$
$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = (1, 0, 4) \cdot (0, 1, 3) = 12$$
$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = (0, 1, 3) \cdot (0, 1, 3) = 10.$$

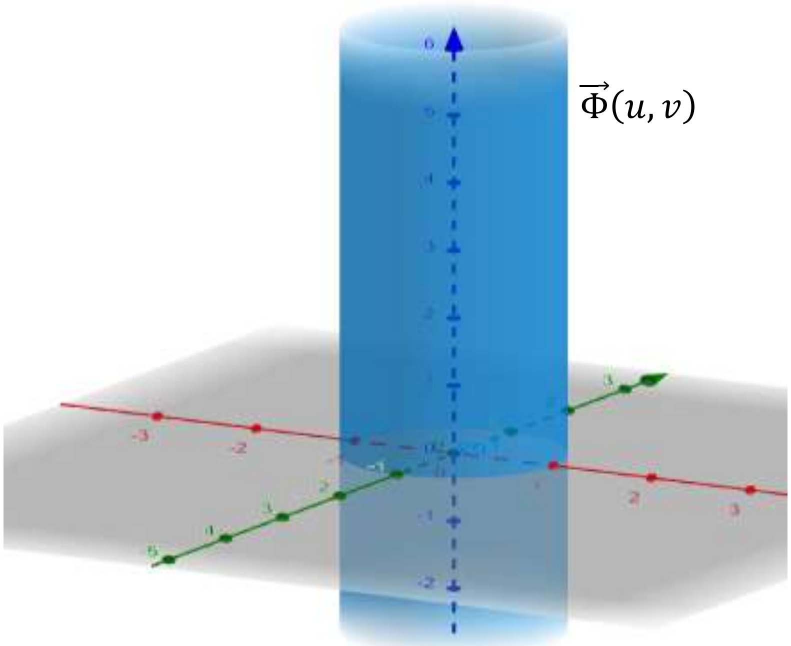
So the first fundamental form is:

$$17du^2 + 24dudv + 10dv^2.$$

Equivalently, the metric tensor is: $g = \begin{pmatrix} 17 & 12 \\ 12 & 10 \end{pmatrix}$.

Ex. Find the first fundamental form for the circular cylinder $x^2 + y^2 = 1$, parametrized by:

$$\vec{\Phi}(u, v) = (\cos u\,, \sin u\,, v), \quad 0 \le u \le 2\pi, \quad v \in \mathbb{R}.$$



$$\vec{\Phi}_u = (-\sin u\,, \cos u\,, 0), \qquad \vec{\Phi}_v = (0, 0, 1)$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = (-\sin u\,, \cos u\,, 0) \cdot (-\sin u\,, \cos u\,, 0) = 1$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = (-\sin u\,, \cos u\,, 0) \cdot (0, 0, 1) = 0$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = (0, 0, 1) \cdot (0, 0, 1) = 1.$$

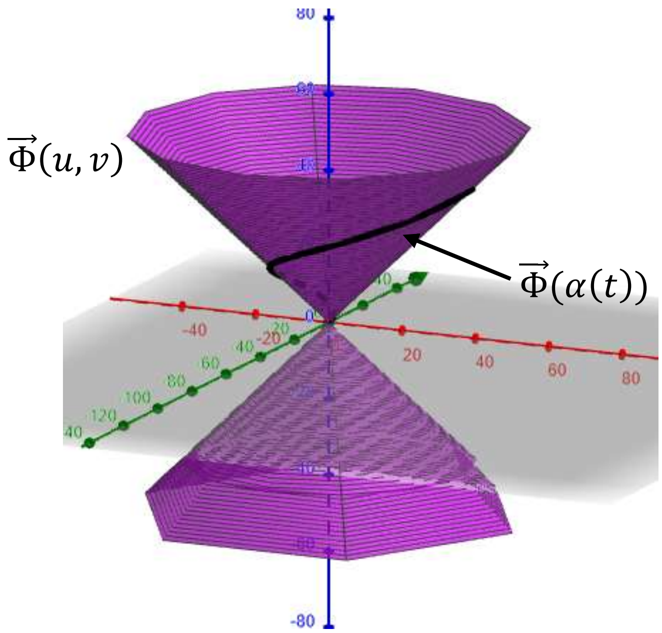
So the first fundamental form is $du^2 + dv^2$.

Equivalently, the metric tensor is:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note: If one parametrizes the x-y plane by $\vec{\Phi}(u,v) = (u,v,0)$ then the first fundamental form is also $du^2 + dv^2$. Thus, the first fundamental form does not uniquely define the surface.

Ex. Find the first fundamental form of $\vec{\Phi}(u,v) = (vcosu, vsinu, v)$; where $0 \le u \le 2\pi$, $v \in \mathbb{R}$, $v \neq 0$, and find the length of the image under $\vec{\Phi}$ of the curve $\alpha(t) = (t, t^2)$; where $0 \le t \le 2\pi$.



$$\vec{\Phi}(u,v) = (vcosu, vsinu, v)$$

$$\vec{\Phi}_u = (-\text{v}\sin u\,, v\cos u\,, 0), \qquad \vec{\Phi}_v = (cosu, sinu, 1).$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = (-\text{v}\sin u\,, v\cos u\,, 0)\cdot(-v\sin u\,, v\cos u\,, 0)$$

$$= v^2\sin^2 u + v^2\cos^2 u = v^2$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = (-\text{v}\sin u\,, \text{v}\cos u\,, 0)\cdot(cosu, sinu, 1) = 0$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = (cosu, sinu, 1)\cdot(cosu, sinu, 1) = 2.$$

So the first fundamental form is $v^2du^2 + 2dv^2$.

Equivalently, the metric tensor is given by:

$$g = \begin{pmatrix} v^2 & 0 \\ 0 & 2 \end{pmatrix}.$$

$$L = \int_a^b (E(u')^2 + 2F(u')(v') + G(v')^2)^{\frac{1}{2}}\, dt.$$

$\alpha(t) = \big(u(t), v(t)\big) = (t, t^2)$;

so $u(t) = t, \quad v(t) = t^2 \quad \Longrightarrow \quad u'(t) = 1, \quad v'(t) = 2t$.

$$L(\overrightarrow{\Phi}(\alpha(t))) = \int_{t=0}^{t=2\pi} (E(u')^2 + G(v')^2)^{\frac{1}{2}}\, dt$$

$$= \int_{t=0}^{t=2\pi} ((v^2(u')^2 + 2(v')^2)^{\frac{1}{2}}\, dt$$

$$= \int_{t=0}^{t=2\pi} ((t^4(1)^2 + 2(2t)^2)^{\frac{1}{2}}\, dt$$

$$= \int_{t=0}^{t=2\pi} (t^4 + 8t^2)^{\frac{1}{2}}\, dt$$

$$= \int_{t=0}^{t=2\pi} t(t^2 + 8)^{\frac{1}{2}}\, dt$$

Let $w = t^2 + 8$; when $t = 0$, $w = 8$

$dw = 2tdt$ when $t = 2\pi$, $w = 4\pi^2 + 8$

$\frac{1}{2}dw = tdt$

$$L(\vec{\Phi}(\alpha(t))) = \frac{1}{2}\int_{w=8}^{w=4\pi^2+8} w^{\frac{1}{2}} dw$$

$$= \frac{1}{2}\left(\frac{2}{3}\right) w^{\frac{3}{2}}\Big|_{w=8}^{w=4\pi^2+8}$$

$$= \frac{1}{3}[(4\pi^2+8)^{\frac{3}{2}} - (8)^{\frac{3}{2}}]$$

$$= \frac{8}{3}\left[(\pi^2+2)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right].$$

Ex. Using the two parametrizations of the upper unit hemisphere below:

1) Find the first fundamental form in each case
2) Find the length of the portion of the great circle starting at $(0,0,1)$ and ending at $(1,0,0)$.

$$\vec{\Phi}(u,v) = ((\cos v)\sin u, (\sin v)\sin u, \cos u); \;\; 0 \le u \le \frac{\pi}{2}, \;\; 0 \le v \le 2\pi$$

$$\vec{\Psi}(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right); \qquad u^2+v^2 \le 1.$$

Let's start with $\vec{\Phi}$.

$$\vec{\Phi}_u = ((\cos v)\cos u, \;\; (\sin v)\cos u, \;\; -\sin u)$$

$$\vec{\Phi}_v = (-(\sin v)\sin u, \;\; (\cos v)\sin u, \;\; 0).$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u$$

$$= ((\cos v)\cos u, (\sin v)\cos u, -\sin u) \cdot ((\cos v)\cos u, (\sin v)\cos u, -\sin u)$$

$$= (\cos^2 v)\cos^2 u + (\sin^2 v)\cos^2 u + \sin^2 u$$

$$= \cos^2 u + \sin^2 u = 1$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v$$
$$= ((\cos v) \cos u\,, (\sin v) \cos u, -\sin u) \cdot (-(\sin v) \sin u\,, (\cos v) \sin u\,, 0)$$
$$= 0.$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v$$
$$= (-(\sin v) \sin u\,, (\cos v) \sin u\,, 0) \cdot (-(\sin v) \sin u\,, (\cos v) \sin u\,, 0)$$
$$= (\sin^2 v) \sin^2 u + (\cos^2 v) \sin^2 u = \sin^2 u.$$

So the first fundamental form is: $du^2 + \sin^2 u\, dv^2$

Equivalently, the metric tensor is: $g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$.

Now for $\vec{\Psi}$:

$$\vec{\Psi}_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right), \qquad \vec{\Psi}_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right)$$

$$E = \vec{\Psi}_u \cdot \vec{\Psi}_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right) \cdot \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right) = 1 + \frac{u^2}{1-u^2-v^2}$$
$$F = \vec{\Psi}_u \cdot \vec{\Psi}_v = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right) \cdot \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right) = \frac{uv}{1-u^2-v^2}$$
$$G = \vec{\Psi}_v \cdot \vec{\Psi}_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right) \cdot \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right) = 1 + \frac{v^2}{1-u^2-v^2}$$

So the first fundamental form is:

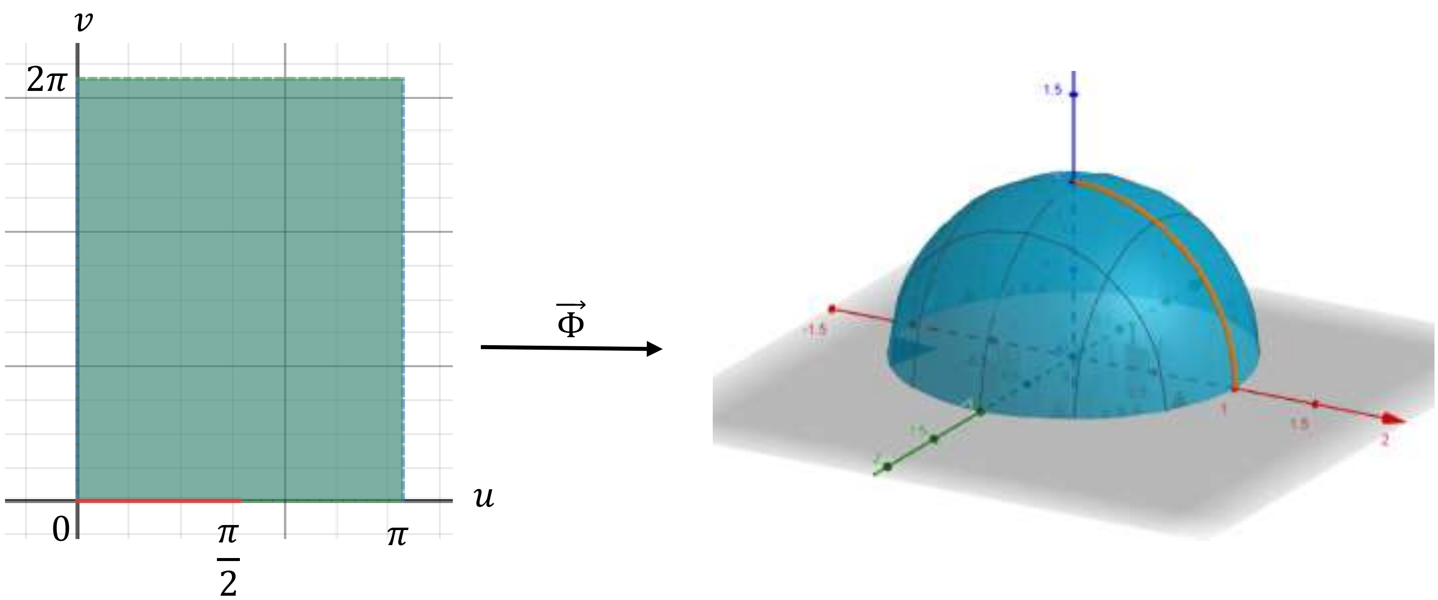
$$\left(1 + \frac{u^2}{1-u^2-v^2}\right) du^2 + \frac{2uv}{1-u^2-v^2}\, du dv + \left(1 + \frac{v^2}{1-u^2-v^2}\right) dv^2$$

Equivalently, the metric tensor is:

$$g = \begin{pmatrix} 1 + \frac{u^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & 1 + \frac{v^2}{1-u^2-v^2} \end{pmatrix}.$$

The portion of the great circle starting at $(0,0,1)$ and ending at $(1,0,0)$ is the image of different curves depending on whether we are using $\overrightarrow{\Phi}$ or $\overrightarrow{\Psi}$.

For $\overrightarrow{\Phi}$, the portion of the great circle is the image of the line segment starting at $u = 0, v = 0$ and ending at $u = \frac{\pi}{2}, v = 0$.



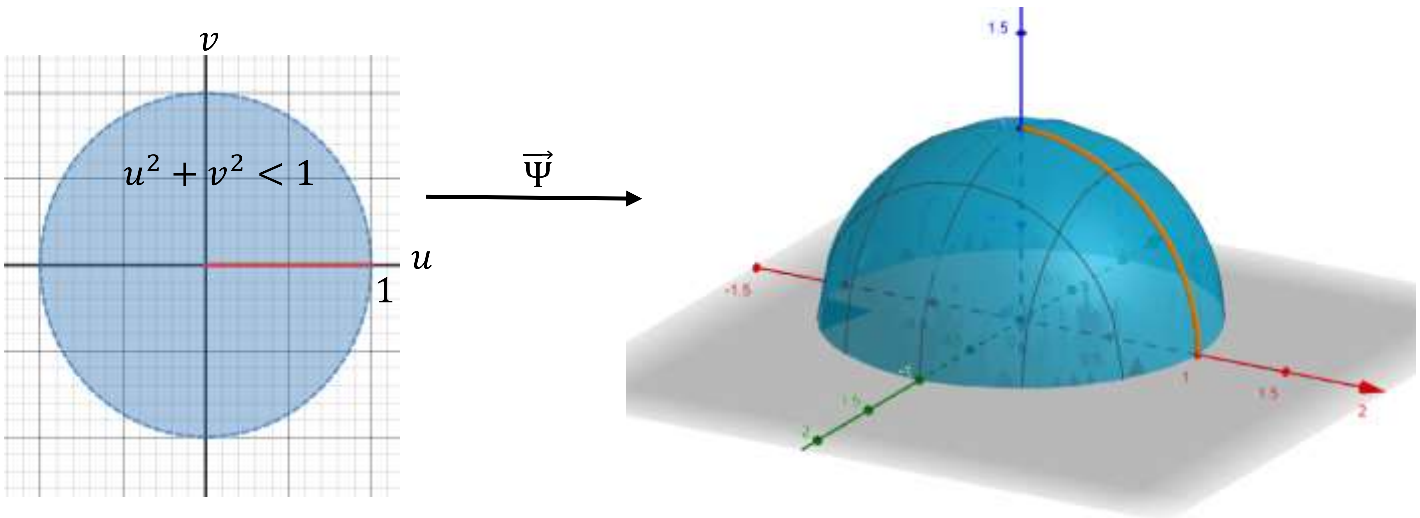
We can parametrize this curve by:

$$\alpha(t) = (t, 0)\,; \quad 0 \leq t \leq \frac{\pi}{2}$$

thus $u(t) = t,\ \ v(t) = 0 \Rightarrow \ \ u'(t) = 1,\ \ v'(t) = 0.$

$$L = \int_0^{\frac{\pi}{2}} (E(u')^2 + 2F(u')(v') + G(v')^2)^{\frac{1}{2}}\ dt = \int_0^{\frac{\pi}{2}} 1\, dt = \frac{\pi}{2}\,.$$

For $\vec{\Psi}$, the portion of the great circle is the image of the line segment starting at $u = 0, v = 0$ and ending at $u = 1, v = 0$.



We can parametrize this curve by:

$$\alpha(t) = (t, 0)\ ; \quad 0 \le t \le 1$$

thus $u(t) = t, \quad v(t) = 0$

$\Rightarrow \quad u'(t) = 1, \ \ v'(t) = 0.$

$$L = \int_0^1 (E(u')^2 + 2F(u')(v') + G(v')^2)^{\frac{1}{2}}\ dt$$

$$= \int_0^1 \left(1 + \frac{t^2}{1-t^2}\right)^{\frac{1}{2}} dt = \int_0^1 \left(\frac{1}{1-t^2}\right)^{\frac{1}{2}} dt = \sin^{-1}(t)\big|_0^1 = \frac{\pi}{2}.$$

Ex. Find the length of the curve given by $x^2 + y^2 = \frac{1}{2}$, $z = \frac{\sqrt{2}}{2}$ on the sphere $x^2 + y^2 + z^2 = 1$, using the parametrization given by:

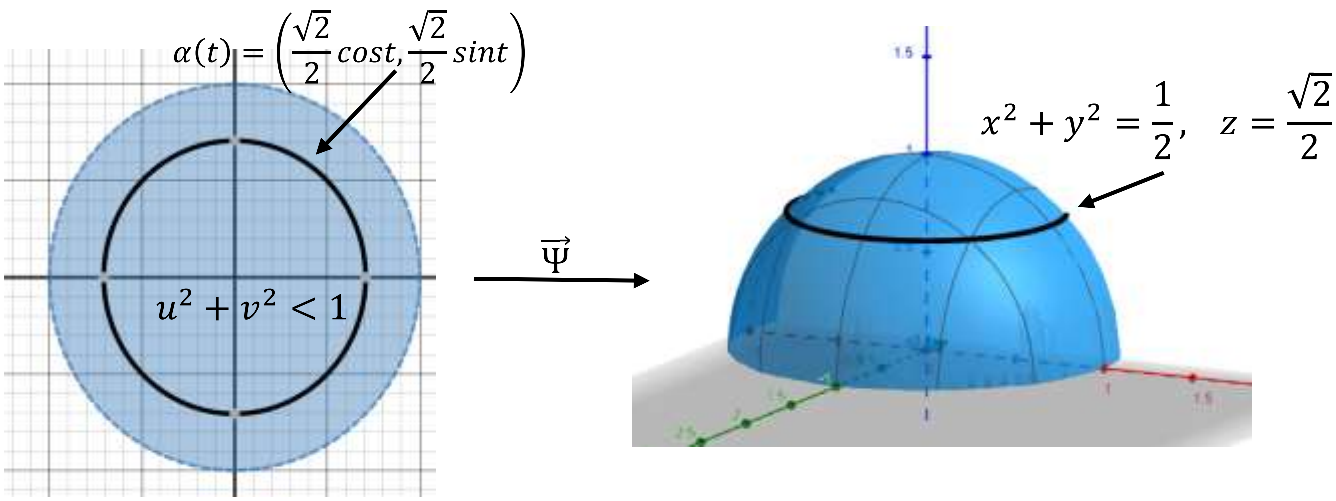
$$\vec{\Psi}(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right); \ \ u^2 + v^2 < 1.$$

First let's parametrize the curve whose image under $\vec{\Psi}$ is

$x^2 + y^2 = \frac{1}{2}$, $z = \frac{\sqrt{2}}{2}$.

In the u-v plane that's the curve $u^2 + v^2 = \frac{1}{2}$.

That's just the curve $\alpha(t) = \left(\frac{\sqrt{2}}{2} cost, \frac{\sqrt{2}}{2} sint\right)$; where $0 \leq t \leq 2\pi$.



Thus: $u(t) = \frac{\sqrt{2}}{2} cost$, $u'(t) = -\frac{\sqrt{2}}{2} sint$

$v(t) = \frac{\sqrt{2}}{2} sint$, $v'(t) = \frac{\sqrt{2}}{2} cost$.

From the previous example we know that:

$$E = 1 + \frac{u^2}{1-u^2-v^2} = 1 + \frac{\frac{1}{2}\cos^2 t}{1-\frac{1}{2}\cos^2 t-\frac{1}{2}\sin^2 t} = 1 + \cos^2 t$$

$$F = \frac{uv}{1-u^2-v^2} = \frac{\left(\frac{\sqrt{2}}{2}cost\right)(\frac{\sqrt{2}}{2}sint)}{1-\frac{1}{2}\cos^2 t-\frac{1}{2}\sin^2 t} = (cost)(sint)$$

$$G = 1 + \frac{v^2}{1-u^2-v^2} = 1 + \frac{\frac{1}{2}\sin^2 t}{1-\frac{1}{2}\cos^2 t-\frac{1}{2}\sin^2 t} = 1 + \sin^2 t.$$

$$L = \int_{t=0}^{t=2\pi} (E(u')^2 + 2F(u')(v') + G(v')^2)^{\frac{1}{2}}\ dt$$

$$E(u')^2 = (1+\cos^2 t)\left(-\frac{\sqrt{2}}{2}sint\right)^2 = \frac{1}{2}(\sin^2 t)(1+\cos^2 t)$$

$$2F(u')(v') = 2(cost)(sint)\left(-\frac{\sqrt{2}}{2}sint\right)\left(\frac{\sqrt{2}}{2}cost\right) = -(\cos^2 t)(\sin^2 t)$$

$$G(v')^2 = (1+\sin^2 t)\left(\frac{\sqrt{2}}{2}cost\right)^2 = \frac{1}{2}(\cos^2 t)(1+\sin^2 t)$$

$$E(u')^2 + 2F(u')(v') + G(v')^2 = \frac{1}{2}$$

$$L = \int_0^{2\pi} \left(\frac{1}{2}\right)^{\frac{1}{2}} dt = \frac{\sqrt{2}}{2} t \Big|_0^{2\pi} = \pi\sqrt{2}\,.$$