Nonhomogeneous Equations: The Methods of Undetermined Coefficients and Variation of Parameters

Recall that if we know a particular solution, y_p, of a nonhomogeneous equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

then the general solution has the form:

$$y = y_c + y_p$$

where y_c is the general solution of the associated homogeneous equation:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0.$$

The method of undetermined coefficients is a way of guessing the form of a particular solution, y_p, based on what $f(x)$ is. If $f(x)$ is simple enough we might be able to do that. For example, if $f(x)$ is a polynomial of degree m then we might guess that y_p is also a polynomial of degree m:

$$y_p = A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0.$$

We could plug y_p into the nonhomogeneous equation and try to solve for the coefficients $A_0, \dots, A_m$. Similarly, if $f(x) = a\cos kx + b\sin kx$ we might guess that:

$$y_p(x) = A\cos kx + B\sin kx.$$

Ex. Solve $y'' - 5y' + 6y = 12x + 8$.

Here $f(x) = 12x + 8$, let's try

$$y_p = Ax + B$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'' - 5y' + 6y = 12x + 8$$

$$0 - 5(A) + 6(Ax + B) = 12x + 8$$

$$-5A + 6Ax + 6B = 12x + 8$$

$$6Ax + (6B - 5A) = 12x + 8$$

$$6A = 12, \quad 6B - 5A = 8$$

$\Longrightarrow A = 2,\ B = 3.$

$y_p = 2x + 3.$

General solution to $y'' - 5y' + 6y = 0$ is:

$r^2 - 5r + 6 = 0$

$(r - 2)(r - 3) = 0$

$r = 2, 3$

$$y_c = c_1 e^{2x} + c_2 e^{3x}.$$

General solution to $y'' - 5y' + 6y = 12x + 8$ is:

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} + 2x + 3.$$

Ex. Find the general solution to $y'' - 3y' - 4y = 2\sin t$.

$f(t) = 2\sin t$, try:

$$y_p = A\sin t + B\cos t$$

$$y'_p = A\cos t - B\sin t$$

$$y''_p = -A\sin t - Bcost$$

$$y'' - 3y' - 4y = 2\sin t$$

$$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$$

$$(-5A + 3B)\sin t + (-5B - 3A)\cos t = 2\sin t$$

$$-5A + 3B = 2$$

$$-3A - 5B = 0.$$

Multiply 2nd equation by $-\frac{5}{3}$: $\quad -5A + 3B = 2$

$$5A + \frac{25}{3}B = 0$$

$$\frac{34}{3}B = 2 \quad \Rightarrow B = \frac{3}{17},\ A = -\frac{5}{17}.$$

So $y_p = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$.

The general solution to $y'' - 3y' - 4y = 0$ is:

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0 \quad \Rightarrow \quad r = 4, -1$$

$$y_c = c_1 e^{-t} + c_2 e^{4t}.$$

The general solution to $y'' - 3y' - 4y = 2\sin t$ is:

$$y = c_1 e^{-t} + c_2 e^{4t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t\ .$$

Ex. Find a particular solution to $y'' - 3y' - 4y = 3e^{2x}$.

$$f(x) = 3e^{2x} \text{ so let's guess:}$$

$$y_p = Ae^{2x}$$

$$y'_p = 2Ae^{2x}$$

$$y''_p = 4Ae^{2x}$$

$$y'' - 3y' - 4y = 3e^{2x}$$

$$4Ae^{2x} - 3(2Ae^{2x}) - 4(Ae^{2x}) = 3e^{2x}$$

$$(4A - 6A - 4A)e^{2x} = 3e^{2x}$$

$$-6Ae^{2x} = 3e^{2x}$$

$$A = -\tfrac{1}{2}$$

$$y_p = -\tfrac{1}{2}e^{2x}.$$

Ex. Find a particular solution to $y'' - 9y = 3e^{3x}$.

$$f(x) = 3e^{3x}, \text{ Try:}$$

$$y_p = Ae^{3x}$$

$$y'_p = 3Ae^{3x}$$

$$y''_p = 9Ae^{3x}$$

$$y'' - 9y = 3e^{3x}$$

$$9Ae^{3x} - 9Ae^{3x} = 3e^{2x}$$

$0 \neq 3e^{3x}$, no matter what A we choose, our guess won't work!

Let's try something else:

$$y_p = Axe^{3x}$$

$$y'_p = A(x(3e^{3x}) + e^{3x}) = 3Axe^{3x} + Ae^{3x}$$

$$y''_p = 3A(x(3e^{3x}) + e^{3x}) + 3Ae^{3x} = 9Axe^{3x} + 6Ae^{3x}$$

Now plugging into the differential equation:

$$(9Axe^{2x} + 6Ae^{2x}) - 9Axe^{2x} = 3e^{3x}$$

$$(9A - 9A)xe^{2x} + 6Ae^{2x} = 3e^{3x}$$

$$A = \frac{1}{2}.$$

$$y_p = \frac{1}{2}xe^{3x}.$$

Method of Undetermined Coefficients (Rule 1)

If no term appearing in $f(x)$ or in any of its derivatives satisfies the related homogeneous equation, then take y_p to be a linear combination of all linearly independent such terms and their derivatives. Then find the coefficients by substituting y_p into the nonhomogeneous equation.

Notice that in the previous example the solution to the homogeneous problem is:

$$y'' - 9y = 0$$

$$r^2 - 9 = 0$$

$$(r - 3)(r + 3) = 0$$

$$r = 3, -3$$

$$y_c = c_1e^{3x} + c_2e^{-3x}$$

So our guess of $y_p = Ae^{3x}$ couldn't work since it's part of the solution to the homogeneous equation.

Ex. Solve the initial value problem:

$$2y'' + 3y' + y = t^2 + 3\sin(t)\,; \quad y(0) = \frac{141}{10},\ \ y'(0) = -\frac{53}{10}.$$

The characteristic equation is: $2r^2 + 3r + 1 = 0$

$$(2r+1)(r+1) = 0,$$

$$r = -\tfrac{1}{2},\ -1$$

$$\Rightarrow \qquad y_c = c_1 e^{-t} + c_2 e^{(-\frac{1}{2}t)}.$$

To find a particular solution we try:

$$y_p = At^2 + Bt + C + Dsin(t) + Ecos(t)$$

$$y_p{}' = 2At + B + Dcos(t) - Esin(t)$$

$$y_p{}'' = 2A - Dsin(t) - Ecos(t)$$

Substituting into the original nonhomogeneous equation we get:

$$2\big(2A - Dsin(t) - Ecos(t)\big) + 3(2At + B + Dcos(t) - Esin(t)) + \big(At^2 + Bt + C + Dsin(t) + Ecos(t)\big) = t^2 + 3\sin(t)$$

$$At^2 + (6A + B)t + (4A + 3B + C) + (-D - 3E)\sin(t) + (3D - E)\cos(t) = t^2 + 3\sin(t).$$

1. $A = 1$
2. $6A + B = 0$
3. $4A + 3B + C = 0$
4. $-D - 3E = 3$
5. $3D - E = 0$

Substituting eq. 1 into eq. 2 we get $B = -6$.

Substituting $A = 1, B = -6$ into eq. 3 we get $C = 14$

Multiplying eq. 4 by 3 and adding it to eq. 5 we get $E = -\frac{9}{10}, \quad D = -\frac{3}{10}$.

Thus we have:

$$y_p = t^2 - 6t + 14 - \frac{3}{10}sin(t) - \frac{9}{10}cos(t)$$

$$y(t) = y_c + y_p$$

$$y = c_1e^{-t} + c_2e^{(-\frac{1}{2}t)} + t^2 - 6t + 14 - \frac{3}{10}sin(t) - \frac{9}{10}cos(t)$$

$$y' = -c_1e^{-t} - \frac{1}{2}c_2e^{(-\frac{1}{2}t)} + 2t - 6 - \frac{3}{10}\cos(t) + \frac{9}{10}\sin(t)$$

$$y(0) = c_1 + c_2 + 14 - \frac{9}{10} = \frac{141}{10} \qquad \Rightarrow \qquad c_1 + c_2 = 1$$

$$y'(0) = -c_1 - \frac{1}{2}c_2 - 6 - \frac{3}{10} = -\frac{53}{10} \qquad \Rightarrow \qquad -c_1 - \frac{1}{2}c_2 = 1$$

$$\Rightarrow \quad c_1 = -3, \quad c_2 = 4$$

$$y(t) = -3e^{-t} + 4e^{-\frac{1}{2}t} + t^2 - 6t + 14 - \frac{3}{10}sin(t) - \frac{9}{10}cos(t).$$

Ex. Find the form of y_p for $y^{(3)} + 4y' = x\cos x + xe^{2x}$.

The characteristic equation is:

$$r^3 + 4r = 0$$
$$r(r^2 + 4) = 0$$
$$r = 0, \quad r = \pm 2i.$$

So, $y_c(x) = c_1 + c_2 \cos 2x + c_3 \sin 2x$.

The function $f(x) = xcosx + xe^{2x}$ and its derivatives involve:

$$x\sin x\,,\;\; x\cos x\,,\;\; \sin x\,,\;\; \cos x\,,\;\; xe^{2x}, \text{ and } e^{2x}$$

So we guess that y_p has the form:

$$y_p(x) = A\,x\sin x + B\,x\cos x + C\sin x + D\cos x + Exe^{2x} + Fe^{2x}.$$

Method of Undetermined Coefficients (Rule 2)

If $a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$ and $f(x)$ can be written as a sum of $P_m(x)e^{rx}\cos kx$ or $P_m(x)e^{rx}\sin kx$, where $P_m(x)$ is an m^{th} degree polynomial, then take:

$$y_p = x^s[A_0 + A_1 x + \cdots + A_m x^m]e^{rx}\cos kx$$
$$+x^s[B_0 + B_1 x + \cdots + B_m x^m]e^{rx}\sin kx.$$

Where s is the smallest nonnegative integer such that no term in y_p duplicates a term in $y_c(x)$.

Ex. Find the form of y_p for $y''' - 3y'' + 3y' - y = (2x-3)e^x$.

The characteristic equation is: $r^3 - 3r^2 + 3r - 1 = 0$

$$(r-1)^3 = 0$$

So $y_c = (c_1 + c_2 x + c_3 x^2)e^x$.

We might guess y_p has the form: $y_p = Ae^x + Bxe^x$, but y_p is part of y_c.

Thus, if we plugged $y_p = Ae^x + Bxe^x$ into the nonhomogeneous eqution, the LHS would be 0 so we couldn't find an A, B so that we get $(2x-3)e^x$.

By Rule 2 we have to multiply $Ae^x + Bxe^x$ by x^3 so neither term is part of $y_c = (c_1 + c_2 x + c_3 x^2)e^x$. The smallest nonnegative integer that will work is $s = 3$. Thus,

$$y_p = x^3(Ae^x + Bxe^x) = Ax^3e^x + Bx^4e^x.$$

Ex. Find a particular solution for $y''' + y'' = 4e^x + 12x^2 + 36x + 18$.

The characteristic equation is $r^3 + r^2 = 0$

$r^2(r+1) = 0, \quad \Rightarrow r = 0$ (double root), $r = -1$.

So $y_c(x) = c_1 + c_2 x + c_3 e^{-x}$.

Our first guess for y_p might be: $y_p = Ae^x + (B + Cx + Dx^2)$.

However, although Ae^x does not duplicate any terms of $y_c = c_1 + c_2 x + c_3 e^{-x}$, $B + Cx + Dx^2$ does.

So we need to multiply $B + Cx + Dx^2$ by x^s so that there is no duplication with $c_1 + c_2x$. The smallest s that will work is $s = 2$.

$$y_p = Ae^x + Bx^2 + Cx^3 + Dx^4$$
$$y_p{}' = Ae^x + 2Bx + 3Cx^2 + 4Dx^3$$
$$y_p{}'' = Ae^x + 2B + 6Cx + 12Dx^2$$
$$y_p{}''' = Ae^x + 6C + 24Dx$$

Now substitute in $y''' + y'' = 4e^x + 12x^2 + 36x + 18$:

$$(Ae^x + 6C + 24Dx) + (Ae^x + 2B + 6Cx + 12Dx^2) = 4e^x + 12x^2 + 36x + 18$$
$$2Ae^x + (6C + 2B) + (24D + 6C)x + 12Dx^2 = 4e^x + 12x^2 + 36x + 18$$

$$2A = 4, \qquad 6C + 2B = 18$$
$$24D + 6C = 36, \qquad 12D = 12.$$

$$\Rightarrow \quad A = 2,\ \ B = 3,\ \ C = 2,\ \ D = 1.$$

So we can write: $y_p = 2e^x + 3x^2 + 2x^3 + x^4$.

General solution:

$$y = y_c + y_p = c_1 + c_2x + c_3e^{-x} + 2e^x + 3x^2 + 2x^3 + x^4.$$

Ex. Find the form of y_p for $y'' + 4y' + 13y = -2e^{-2x}\sin 3x$.

The characteristic equation is $r^2 + 4r + 13 = 0$.

$$\Rightarrow \qquad r = -2 \pm 3i.$$

$$y_c = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x).$$

This is the same as the first guess we might make at y_p

$$y_p = e^{-2x}(A\cos 3x + B\sin 3x).$$

To eliminate duplication, multiply by x:

$$y_p = e^{-2x}(Ax\cos 3x + Bx\sin 3x).$$

Variation of Parameters

For an equation like $y'' + y = \cot x$, we can't use undetermined coefficients because $f(x) = \cot x$ has infinitely many linearly independent derivatives.

Variation of parameters is a method that, in principle, can be used to solve any nonhomogeneous linear differential equation of the form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

however, the method can give rise to integrals that could be difficult to evaluate.

Let's try to solve $y'' + P(x)y' + Q(x)y = f(x)$, with

$y_c = c_1 y_1(x) + c_2 y_2(x)$.

We want to find two functions, $u_1(x), u_2(x)$ such that:

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$.

So we have to find $u_1(x), u_2(x)$ so that y_p satisfies:

$y'' + P(x)y' + Q(x)y = f(x)$.

Since we have two functions, $u_1(x), u_2(x)$, we can satisfy two conditions. The first condition is that:

$$y_p'' + P(x)y_p' + Q(x)y_p = f(x).$$

We will use the second condition to simplify the calculation.

Since $y_p{}' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$ the second condition will be:

$$u_1'y_1 + u_2'y_2 = 0$$

so that $u_1{}''$ and $u_2{}''$ don't appear when we take $y_p{}''$.

So we have:

$$y_p' = u_1y_1' + u_2y_2'$$

$$y_p'' = (u_1y_1'' + u_2y_2'') + (u_1'y_1' + u_2'y_2').$$

Both y_1 and y_2 satisfy the homogeneous equation:

$$y'' + P(x)y' + Q(x)y = 0$$

so

$$y_1'' = -Py_1' - Qy_1$$

$$y_2'' = -Py_2' - Qy_2.$$

Thus we have,

$$y_p'' = (u_1'y_1' + u_2'y_2') - P(u_1y_1' + u_2y_2') - Q(u_1y_1 + u_2y_2)$$

$$y_p'' = (u_1'y_1' + u_2'y_2') - Py_p' - Qy_p$$

$$\Rightarrow \qquad y_p'' + Py_p' + Qy_p = u_1'y_1' + u_2'y_2'.$$

Since y_p must satisfy:

$$y'' + Py' + Qy = f(x)$$

$$u_1'y_1' + u_2'y_2' = f(x).$$

So we have a system of equations:

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = f(x).$$

Notice $W(y_1, y_2) = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$

Then solving these simultaneous equations for u'_1 and u'_2 we get:

$$u_1' = \frac{1}{W(y_1, y_2)}(-y_2\, f(x))$$

$$u_2' = \frac{1}{W(y_1, y_2)}\big(y_1\, f(x)\big).$$

Since $y_p = u_1y_1 + u_2y_2$,

$$y_p = -y_1 \int \frac{y_2\, f(x)}{W(y_1, y_2)}\, dx + y_2 \int \frac{y_1\, f(x)}{W(y_1, y_2)}\, dx.$$

Ex. Find a particular solution for $y'' + y = \cot x$.

First find $y_c = c_1 y_1 + c_2 y_2$.

The characteristic equation is $r^2 + 1 = 0$, so $r = \pm i$.

$$\Rightarrow \qquad y_c = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x \qquad y_2 = \sin x \qquad (*)$$

$$y_1' = -\sin x \qquad y_2' = \cos x.$$

$$W(y_1, y_2) = \det\begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = 1.$$

$$y_p = -y_1 \int \frac{y_2\, f(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1\, f(x)}{W(y_1, y_2)}\ dx$$

$$= -cosx \int sinx(cotx)dx + sinx \int cosx(cotx)dx$$

$$= -cosx \int cosxdx + sinx \int \frac{\cos^2 x}{sinx} dx$$

$$\int \frac{\cos^2 x}{sinx} dx = \int \frac{1-\sin^2 x}{sinx} dx = \int (cscx - sinx)dx$$

$$= -\ln|cscx + cotx| + cosx + C$$

So we have:

$$y_p = -cosx(sinx) - (sinx)\ln|cscx + cotx| + sinx(cosx)$$

$$= -(sinx)(\ln|cscx + cotx|).$$