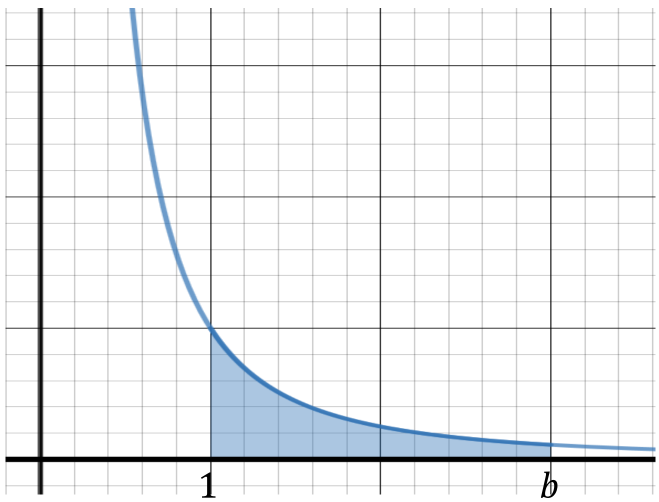
Improper Integrals

Up to this point we have only discussed integrals of bounded functions on a finite interval. We will now investigate integrals of functions over unbounded intervals and integrals of functions with infinite discontinuities.

Infinite Intervals

So far we know how to evaluate $\int_1^b \frac{1}{x^2}\,dx,\ \ b \geq 1.$



$$\int_1^b \frac{1}{x^2}\,dx = \frac{-1}{x}\bigg|_1^b = \frac{-1}{b} + 1 = 1 - \frac{1}{b}.$$

A natural way to define $\int_1^\infty \frac{1}{x^2}\,dx$ is to say:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

Definitions of integrals over infinite intervals:

1. If f is continuous on $[a, \infty)$, then:

$$\int_{a}^{\infty} \boldsymbol{f(x)dx} = \lim_{b\to\infty} \int_{a}^{b} \boldsymbol{f(x)dx}$$

2. If f is continuous on $(-\infty, b]$, then:

$$\int_{-\infty}^{b} \boldsymbol{f(x)dx} = \lim_{a\to-\infty} \int_{a}^{b} \boldsymbol{f(x)dx}$$

3. If f is continuous on $(-\infty, \infty)$, then:

$$\int_{-\infty}^{\infty} \boldsymbol{f(x)dx} = \int_{-\infty}^{c} \boldsymbol{f(x)dx} + \int_{c}^{\infty} \boldsymbol{f(x)dx}$$

where c is any real number.

In the first two cases, if the limit exists, we say the improper integral **converges**, otherwise we say that the improper integral **diverges**. In the third case, the improper integral on the left diverges if either improper integral on the right diverges. The improper integral on the left converges only if both improper integrals on the right converge.

Ex. Decide if the following integral converges or diverges: $\int_1^\infty \frac{1}{x^2}\,dx$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \frac{-1}{x}\bigg|_1^b = \lim_{b\to\infty} \left(\frac{-1}{b} + 1\right) = 1$$

So $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

Ex. Evaluate $\int_1^\infty \frac{1}{x} dx$.

$$\int_1^\infty \frac{1}{x} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to\infty} lnx\Big|_1^b = \lim_{b\to\infty}(lnb - ln1)$$

$$= \infty$$

So the integral diverges.

Ex. Evaluate $\int_{-\infty}^0 xe^x dx$.

$$\int_{-\infty}^0 xe^x dx = \lim_{a\to-\infty} \int_a^0 xe^x dx$$

Now integrate by parts:

Let $u = x \qquad v = e^x$

$du = dx \qquad dv = e^x dx$

$$= \lim_{a\to-\infty} \left(xe^x|_a^0 - \int_a^0 e^x dx \right)$$

$$= \lim_{a\to-\infty} (0 - ae^a - e^x|_a^0) = \lim_{a\to-\infty} \left(-ae^a - (e^0 - e^a)\right)$$

$$= \lim_{a\to-\infty} (-ae^a - 1 + e^a);$$

$\lim_{a\to-\infty} ae^a = \lim_{a\to-\infty} \frac{a}{e^{-a}} = \lim_{a\to-\infty} \frac{-1}{e^{-a}}$ (L'Hospital's Rule)

$= 0.$

So we have:

$$\int_{-\infty}^0 xe^x dx = \lim_{a\to-\infty} (-ae^a - 1 + e^a) = -1.$$

Ex. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

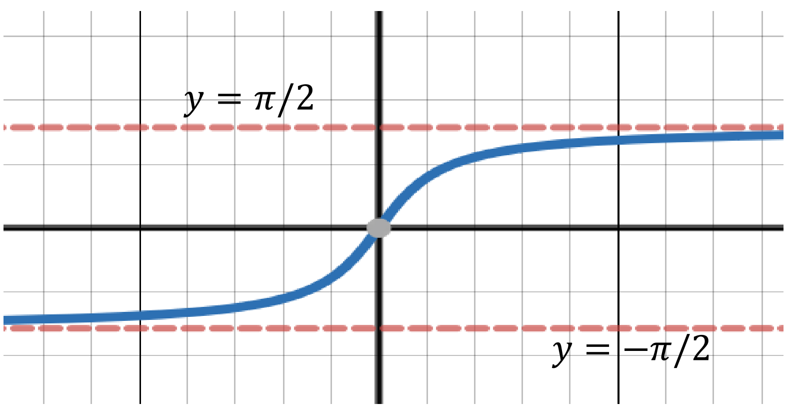
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} tan^{-1}x\Big|_{a}^{0} + \lim_{b\to\infty} tan^{-1}x\Big|_{0}^{b}$$

$$= \lim_{a\to-\infty} (tan^{-1}0 - tan^{-1}a) + \lim_{b\to\infty} (tan^{-1}b - tan^{-1}0)$$

Recall that the graph of $y = tan^{-1}x$ looks like:



$$\lim_{x\to\infty} tan^{-1}x = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} tan^{-1}x = \frac{-\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \left(0 + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right) = \pi.$$

Ex. Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx.$

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}} dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}} dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{e^x}{1+e^{2x}} dx + \lim_{b \to \infty} \int_{0}^{b} \frac{e^x}{1+e^{2x}} dx$$

Let $u = e^x$; $\quad x = a \Rightarrow u = e^a$; $\quad x = 0 \Rightarrow u = e^0 = 1$

$du = e^x dx \quad x = b \Rightarrow u = e^b$;

$$= \lim_{a \to -\infty} \int_{u=e^a}^{u=1} \frac{1}{1+u^2} du + \lim_{b \to \infty} \int_{u=1}^{u=e^b} \frac{1}{1+u^2} du$$

$$= \lim_{a \to -\infty} tan^{-1}u \Big|_{u=e^a}^{u=1} + \lim_{b \to \infty} tan^{-1}u \Big|_{u=1}^{u=e^b}$$

$$= \lim_{a \to -\infty} (tan^{-1}1 - tan^{-1}e^a) + \lim_{b \to \infty} (tan^{-1}(e^b) - tan^{-1}1)$$

$$= \left(\frac{\pi}{4} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2}.$$

Ex. For what values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

If $p \neq 1$ then:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\frac{x^{-p+1}}{-p+1}\bigg|_1^b$$

$$= \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)$$

If $-p+1 > 0 \Rightarrow \quad 1 > p$, then:

$$\lim_{b\to\infty} b^{-p+1} = \infty$$

If $-p+1 < 0 \Rightarrow \quad 1 < p$, then:

$$\lim_{b\to\infty} b^{-p+1} = 0$$

So the integral converges if $p > 1$ and diverges if $p \leq 1$,

since we already saw in a previous example that when $p = 1$,

$\int_1^\infty \frac{1}{x}\,dx$ diverges.

Note:

$\int_1^\infty \frac{1}{x^p}\,dx$ will be important when we discuss infinite series.

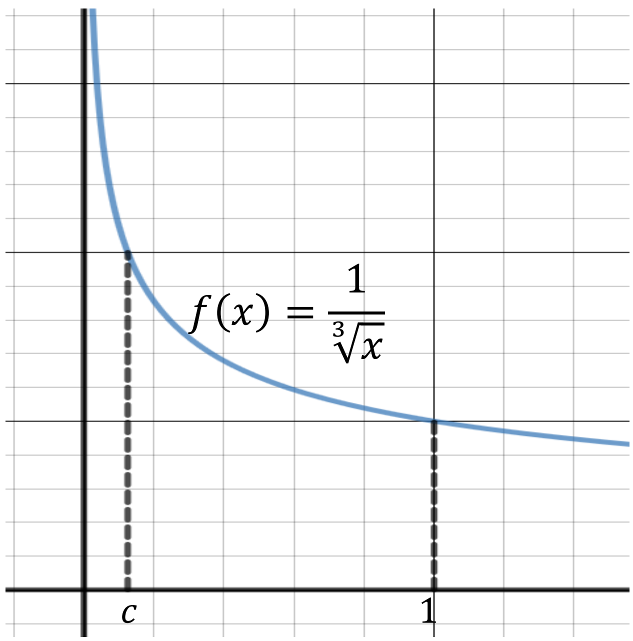
Discontinuous Integrands

$f(x) = \frac{1}{\sqrt[3]{x}}$ is discontinuous at $x = 0$ because:

$$\lim_{x \to 0^+} \frac{1}{\sqrt[3]{x}} = \infty$$

However, if $0 < c < 1$, then:

$$\int_c^1 \frac{1}{\sqrt[3]{x}} dx = \int_c^1 x^{\frac{-1}{3}} dx = \frac{3}{2} x^{\frac{2}{3}} \Big|_c^1 = \frac{3}{2} - \frac{3c^{2/3}}{2}$$



A natural way to define $\int_0^1 \frac{1}{\sqrt[3]{x}} dx$ is:

$$\int_0^1 \frac{1}{\sqrt[3]{x}} dx = \lim_{c \to 0^+} \int_c^1 \frac{1}{\sqrt[3]{x}} dx = \lim_{c \to 0^+} \left(\frac{3}{2} - \frac{3}{2} c^{\frac{2}{3}}\right) = \frac{3}{2}.$$

Definitions for improper integrals with infinite discontinuities:

1. If f is continuous on $[a, b)$ and has an infinite discontinuity at b, then:

$$\int_{a}^{b} f(x)dx = \lim_{c \to b^-} \int_{a}^{c} f(x)dx$$

2. If f is continuous on $(a, b]$ and has an infinite discontinuity at a, then

$$\int_{a}^{b} f(x)dx = \lim_{c \to a^+} \int_{c}^{b} f(x)dx$$

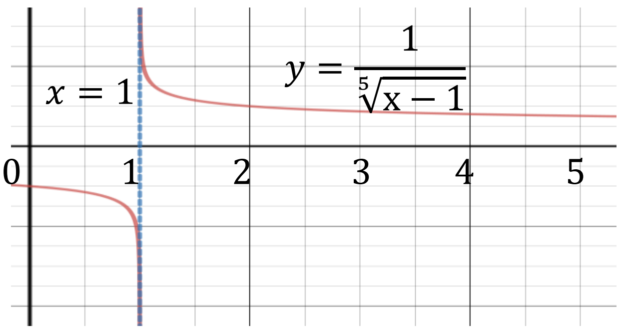
3. If f is continuous on $[a, b]$ except for some c in $[a, b]$ at which f has an infinite discontinuity, then:

$$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$$

In the first two cases, if the limit exists, we say the improper integral converges, otherwise we say it diverges. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverge. If both improper integrals on the right converge, then we say the improper integral on the left converges.

Ex. Find $\int_1^5 \frac{1}{\sqrt[5]{x-1}}dx$.

$\frac{1}{\sqrt[5]{x-1}}$ has a vertical asymptote at $x = 1$



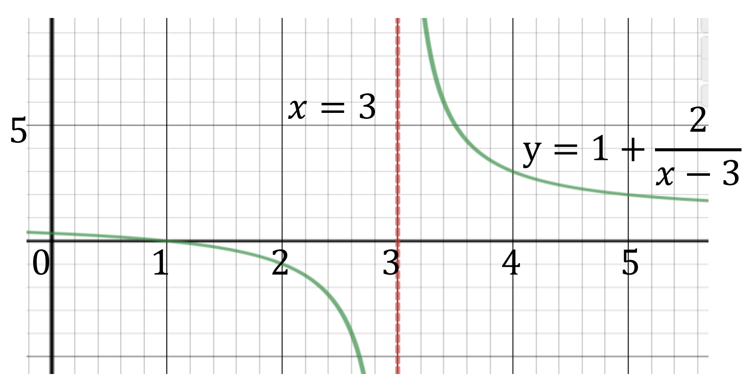
$$\int_1^5 \frac{1}{\sqrt[5]{x-1}}dx = \lim_{c \to 1^+} \int_c^5 (x-1)^{-\frac{1}{5}}dx = \lim_{c \to 1^+} \frac{5}{4}(x-1)^{\frac{4}{5}}\Big|_c^5$$

$$= \lim_{c \to 1^+} \frac{5}{4}\left(4^{\frac{4}{5}} - (c-1)^{\frac{4}{5}}\right) = \frac{5}{4}\left(4^{\frac{4}{5}}\right) = \frac{5}{\sqrt[5]{4}}.$$

So the integral converges.

Ex. Evaluate $\int_1^5 (1 + \frac{2}{x-3})dx$.

$y = 1 + \frac{2}{x-3}$ has a vertical asymptote at $x = 3$



$$\int_1^5 (1+\frac{2}{x-3})dx = \lim_{c\to 3^-}\int_1^c 1+\frac{2}{x-3}\,dx + \lim_{c\to 3^+}\int_c^5 1+\frac{2}{x-3}\,dx$$

$$= \lim_{c\to 3^-}(\mathrm{x}+2\ln|x-3|)\Big|_1^c + \lim_{c\to 3^+}(\mathrm{x}+2\ln|x-3|)\Big|_c^5$$

$$\lim_{C\to 3^-} 2\ln|x-3|\Big|_1^c = \lim_{c\to 3^-} 2[\ln|c-3| - \ln|1-3|] = -\infty$$

$$\lim_{C\to 3^+} \ln|x-3|\Big|_c^5 = \lim_{c\to 3^+} 2[\ln|5-3| - \ln|c-3|] = +\infty$$

So both $\int_1^3 (1+\frac{2}{x-3})dx$ and $\int_3^5 (1+\frac{2}{x-3})dx$ diverge.

Thus, $\int_1^5 (1+\frac{2}{x-3})dx$ diverges.

You <u>must</u> separate an integral with an interior infinite discontinuity into two improper integrals. Look what happens if we ignore the infinite discontinuity:

$$\int_1^5 (1+\frac{2}{x-3})dx = x + 2ln|x-3|\Bigg|_1^5$$

$$= (5+2ln|5-3|) - (1+2ln|1-3|)$$

$$= (5+2ln2) - (1+2ln2) = 4.$$

We arrive at a solution, which is **wrong**.

Ex. For what values of p does $\int_0^1 \frac{1}{x^p}\,dx$ converge?

If $p \neq 1$ then:

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \int_c^1 x^{-p}\,dx = \lim_{c\to 0^+} \frac{x^{-p+1}}{-p+1}\bigg|_c^1$$

$$= \lim_{c\to 0^+} \frac{1}{-p+1}\left(1 - c^{-p+1}\right)$$

If $-p+1 > 0 \quad \Rightarrow \quad p < 1$, then:

$$\lim_{c\to 0^+} c^{-p+1} = 0$$

If $-p+1 < 0 \quad \Rightarrow \quad p > 1$, then:

$$\lim_{c\to 0^+} c^{-p+1} = \infty$$

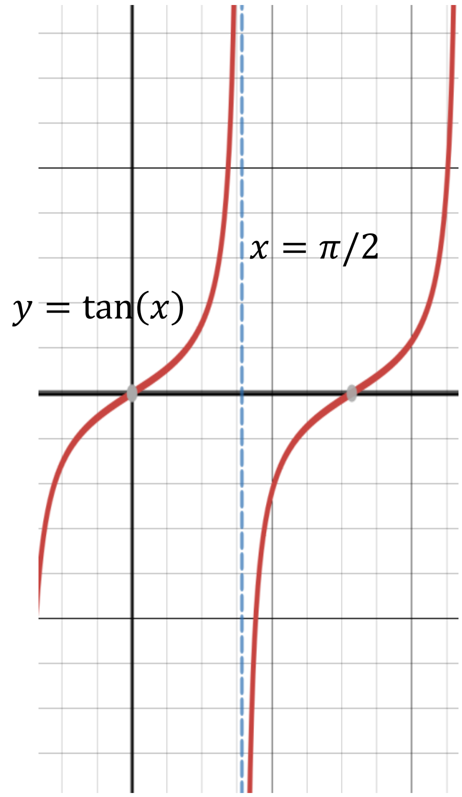
If $p = 1$, then:

$$\lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \ln|x|\bigg|_c^1$$

$$= \lim_{c\to 0^+} \left(ln|1| - ln|c|\right) = \infty.$$

So $\int_0^1 \frac{1}{x^p}\,dx$ converges when $p < 1$ and diverges when $p \geq 1$.

Ex. Evaluate $\int_0^{\frac{\pi}{2}} \tan x \, dx$.



$\tan x = \dfrac{\sin x}{\cos x}$
and has an infinite discontinuity at $\frac{\pi}{2}$.

$$\int_0^{\frac{\pi}{2}} \tan x \, dx = \lim_{c \to \frac{\pi}{2}^-} \int_0^c \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$ $\quad x = 0 \Rightarrow u = 1$

$-du = \sin x$ $\quad x = c \Rightarrow u = \cos c$

$$= \lim_{c \to \frac{\pi}{2}^-} - \int_{u=1}^{u=\cos c} \frac{du}{u}$$

$$= \lim_{C \to \frac{\pi}{2}^-} -\ln |u| \Big|_{u=1}^{u=\cos c}$$

$$= \lim_{C \to \frac{\pi}{2}^-} [-\ln|\cos c| + \ln|1|]$$

$$= \infty.$$

So the integral diverges.

Here's an example of an improper integral where the interval of the integral is unbounded and the function has an infinite discontinuity.

Ex. Evaluate $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$.

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$$

$$= \lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt{x}(x+1)} + \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}(x+1)}$$

Notice that if we let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$.
We can now see that:

$$\int \frac{dx}{\sqrt{x}(x+1)} = 2\int \frac{du}{1+u^2} = 2\tan^{-1}(\sqrt{x}) + C$$

So we can write:

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \lim_{c\to 0^+} 2\tan^{-1}(\sqrt{x})\Big|_c^1 + \lim_{b\to\infty} 2\tan^{-1}(\sqrt{x})\Big|_1^b$$

$$= \lim_{c\to 0^+} 2\left(\tan^{-1}(1) - \tan^{-1}(\sqrt{c})\right) + \lim_{b\to\infty} 2\left(\tan^{-1}(\sqrt{b}) - \tan^{-1}(1)\right)$$

$$= 2\left(\frac{\pi}{4} - 0\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Comparison Test for Improper Integrals

Sometimes it's enough to know whether an improper integral converges or diverges (rather than knowing the actual value if it converges).

Comparison Theorem: Suppose that f and g are continuous functions with

$$f(x) \geq g(x) \geq 0 \text{ for } x \geq a.$$

a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

One way to remember this theorem is to note that if $f(x) \geq g(x) \geq 0$, then:

$$\int_a^\infty f(x)dx \geq \int_a^\infty g(x)dx \geq 0.$$

Thus, we can conclude:

If $\int_a^\infty f(x)dx$ is finite, then $\int_a^\infty g(x)dx$ is finite.

If $\int_a^\infty g(x)dx$ is infinite, then $\int_a^\infty f(x)dx$ is infinite.

This is not a proof, just a way to remember the theorem

Notice that the theorem doesn't allow us to conclude anything about:

$\int_a^\infty g(x)dx$ if $\int_a^\infty f(x)dx$ is divergent

or

$\int_a^\infty f(x)dx$ if $\int_a^\infty g(x)dx$ is convergent.

Ex. Use the comparison theorem to show that $\int_1^\infty \frac{x^2}{x^5+3}\,dx$ is convergent.

Notice that for $x \geq 1$:

$$\frac{x^2}{x^5+3} \leq \frac{x^2}{x^5} = \frac{1}{x^3}$$

Let $g(x) = \frac{x^2}{x^5+3}$ and $f(x) = \frac{1}{x^3}$.

Since $0 \leq g(x) \leq f(x)$ for $x \geq 1$ and $\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^3}\,dx$ converges because $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$.

Thus, $\int_1^\infty g(x)dx = \int_1^\infty \frac{x^2}{x^5+3}\,dx$ converges by the comparison test.

Ex. Use the comparison theorem to show $\int_1^\infty \frac{1+(\ln x)^2}{x+1}\,dx$ diverges.

Notice that for $x \geq 1$:

$$\frac{1+(\ln x)^2}{x+1} \geq \frac{1}{x+1} \geq 0$$

Let $f(x) = \frac{1+(\ln x)^2}{x+1}$ and $g(x) = \frac{1}{x+1}$.

$$\int_1^\infty \frac{1}{x+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+1}\,dx = \lim_{b\to\infty} \ln|x+1|\Big|_1^b$$

$$= \lim_{b\to\infty} (\ln|b+1| - \ln|2|) = \infty$$

So $\int_1^\infty \frac{1+(\ln x)^2}{x+1}\,dx$ diverges by the comparison theorem.

Ex. Determine if $\int_0^{\infty} \frac{Tan^{-1}x}{2+e^x}\,dx$ converges or diverges.

Notice that if $x \geq 0$ then $0 \leq \tan^{-1} x \leq \frac{\pi}{2}$.

Thus we have:

$$0 \leq \frac{Tan^{-1}x}{2+e^x} \leq \frac{\pi}{2}\left(\frac{1}{2+e^x}\right) \leq \frac{\pi}{2}\left(\frac{1}{e^x}\right) = \frac{\pi}{2}e^{-x}.$$

Let $g(x) = \frac{Tan^{-1}x}{2+e^x}$ and $f(x) = \frac{\pi}{2}e^{-x}$.

$$0 \leq \int_0^{\infty} \frac{Tan^{-1}x}{2+e^x}\,dx \leq \int_0^{\infty} \frac{\pi}{2}e^{-x}dx.$$

$$\int_0^{\infty} \frac{\pi}{2}e^{-x}dx = \frac{\pi}{2}\lim_{b\to\infty}\int_0^{b} e^{-x}dx$$

$$= \frac{\pi}{2}\lim_{b\to\infty} - e^{-x}\Big|_0^b$$

$$= \frac{\pi}{2}\lim_{b\to\infty}[-(e^{-b} - 1)]$$

$$= \frac{\pi}{2}\,.$$

So $\int_0^{\infty} \frac{Tan^{-1}x}{2+e^x}\,dx$ converges.