The Lebesgue Integral $\int_E f$: f Bounded, $m(E) < \infty$

Note: From now on integration will mean Lebesgue integration unless otherwise specified.

Let $\psi = \sum_{i=1}^{n} a_i \chi_{E_i}$ on E, where $E_i = \psi^{-1}(a_i) = \{x \in E \,|\, \psi(x) = a_i\}$

Be a simple function ($a_i{}'s$ are distinct and $\{E_i\}$ disjoint).

Def. For a simple function ψ defined on a set of finite measure E, define

$$\boldsymbol{\int_E \psi = \sum_{i=1}^{n} a_i (m(E_i))}.$$

Notice that this definition of $\int_E \psi$ allows us to evaluate the following integral.

Ex. Let $f(x) = 1$ if $x \in \mathbb{Q} \cap [0,1] = E_1$

$= 0$ if $[0,1] \sim E_1$.

Evaluate $\int_{[0,1]} f$.

Let $E_1 = \mathbb{Q} \cap [0,1]$ and $E_2 = [0,1] \sim E_1$, then we can write:

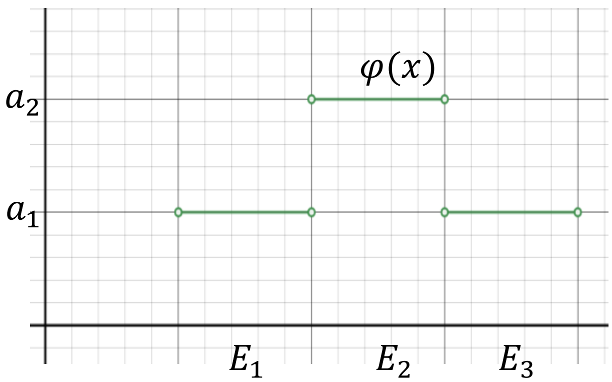
$$f(x) = 1(\chi_{E_1}) + 0(\chi_{E_2}) = \chi_{E_2}.$$

Thus, $\int_{[0,1]} f = 1\left(m(\chi_{E_1})\right) = 0.$

Lemma: Let $\{E_i\}_{i=1}^n$ be a finite disjoint collection of measurable subsets of a set of finite measure E. For $1 \le i \le n$, let a_i be a real number. If

$\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ on E then $\int_E \varphi = \sum_{i=1}^n a_i (m(E_i))$.

Proof: The issue here is that $\{a_i\}$ may not be distinct (i.e., φ is not written in canonical form). If we rewrite φ in canonical form the result readily follows.



$\varphi(x) = a_1\chi_{E_1} + a_2\chi_{E_2} + a_1\chi_{E_3}$;

Let $E' = E_1 \cup E_3$ then we can

Write φ in canonical form:

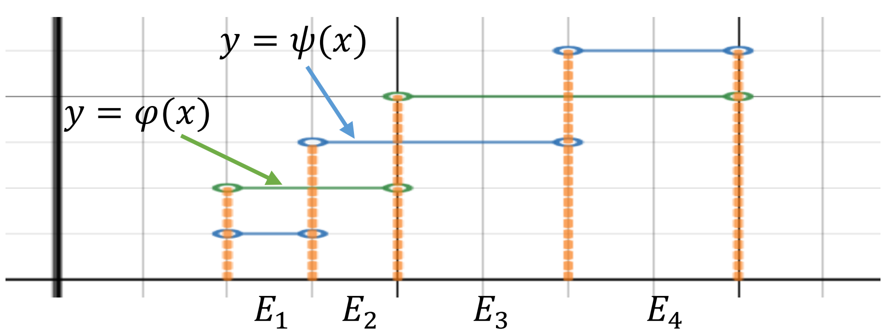
$\varphi(x) = a_1\chi_{E'} + a_2\chi_{E_2}$

Prop. Let φ and ψ be simple function defined on a set of finite measure E. Then

1. for $\alpha, \beta \in \mathbb{R}$ $\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi$.
2. if $\varphi \le \psi$ on E then

$$\int_E \varphi \le \int_E \psi.$$

Proof: Since φ and ψ are simple we can find a finite disjoint collection of sets $\{E_i\}_{i=1}^n$ such that φ and ψ are constant on each E_i and $E = \bigcup_{i=1}^n E_i$.



For each $1 \leq i \leq n$ let: $\varphi(x) = a_i$ and $\psi(x) = b_i$ for $x \in E_i$.

By the preceding lemma:

$$\int_E \varphi = \sum_{i=1}^{n} a_i(m(E_i)) \text{ and } \int_E \psi = \sum_{i=1}^{n} b_i(m(E_i)).$$

The simple function $\alpha\varphi + \beta\psi$ has:

$$(\alpha\varphi + \beta\psi)(x) = \alpha a_i + \beta b_i \quad \text{for } x \in E_i.$$

Again by the preceding lemma we have:

$$\begin{aligned} \int_E (\alpha\varphi + \beta\psi) &= \sum_{i=1}^{n} (\alpha a_i + \beta b_i)\big(m(E_i)\big) \\ &= \alpha \sum_{i=1}^{n} (a_i)\big(m(E_i)\big) + \beta \sum_{i=1}^{n} (b_i)\big(m(E_i)\big) \\ &= \alpha \int_E \varphi + \beta \int_E \psi. \end{aligned}$$

For the second part, let $g = \psi - \varphi \geq 0$.

By the first part: $\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) = \int_E g \geq 0$

since $g \geq 0$.

Thus $\int_E \psi \geq \int_E \varphi$.

Notice that a step function is an example of a simple function (where E_i is an interval). Since the measure of an interval is its length, we can see that the definition of **the Lebesgue integral and Riemann integral agree for step functions.**

Let f be a bounded real valued function defined on a set of finite measure E. We define the lower and upper Lebesgue integrals of f by:

Lower Lebesgue Integral of *f*$=\sup\{\int_E \varphi |\ \varphi\ is\ simple\ and\ \varphi \leq f\}$

Upper Lebesgue Integral of *f*$=\inf\left\{\int_E \varphi |\ \varphi\ is\ simple\ and\ \varphi \geq f\right\}$.

Since f is bounded, by the monotonicity property (if $\varphi \leq \psi$ then $\int_E \varphi \leq \int_E \psi$), the lower and upper integrals are finite and the upper integral is always at least as large as the lower integral.

Def. A bounded function f on a domain E of finite measure is said to be **Lebesgue integrable over $\boldsymbol{E}$** if its upper and lower Lebesgue integral over E are equal. That common value is called the Lebesgue integral of f over E, denoted by $\int_E f$.

Theorem: Let f be a bounded function defined on a closed, bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.

Proof: Each step function is a simple function and the Riemann and Lebesgue integrals agree for step functions.

Theorem: Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.

Proof: Let $n \in \mathbb{Z}^+$.

By the Simple Approximation Theorem with $\epsilon = \frac{1}{n}$ there are two simple functions φ_n, ψ_n on E with

$$\varphi_n \le f \le \psi_n \quad \text{and} \quad 0 \le \psi_n - \varphi_n \le \frac{1}{n}, \quad \text{on } E.$$

Thus we have:

$$0 \le \int_E \psi_n - \int_E \varphi_n = \int_E (\psi_n - \varphi_n) \le \frac{1}{n}(m(E)).$$

Now notice that:

$$0 \le \inf\left\{\int_E \varphi \mid \varphi \text{ is simple and } \varphi \ge f\right\} - \sup\left\{\int_E \varphi \mid \varphi \text{ is simple and } \varphi \le f\right\}$$

$$\le \int_E \psi_n - \int_E \varphi_n \le \frac{1}{n}(m(E)).$$

Now let n go to ∞,

so the upper and lower integrals are equal and f is integrable.

Theorem: Let f and g be bounded measurable function on a set of finite measure E. Then

1. for $\alpha, \beta \in \mathbb{R}$, $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.
2. if $f \leq g$ on E then

$$\int_E f \leq \int_E g.$$

Proof: $\alpha f + \beta g$ is a bounded measurable function on E because f and g are, hence it's integrable.

First let's show $\int_E \alpha f = \alpha \int_E f$.

If $\alpha > 0$ then

$$\int_E \alpha f = \inf_{\psi \geq \alpha f} \int_E \psi = (\alpha) \inf_{\frac{\psi}{\alpha} \geq f} \int_E \frac{\psi}{\alpha} = \alpha \int_E f.$$

If $\alpha < 0$ then

$$\int_E \alpha f = \inf_{\psi \geq \alpha f} \int_E \psi = (\alpha) \sup_{\frac{\psi}{\alpha} \leq f} \int_E \frac{\psi}{\alpha} = \alpha \int_E f .$$

To establish linearity we just need to show $\int_E (f + g) = \int_E f + \int_E g$.

Let ψ_1 and ψ_2 be simple functions with $f \leq \psi_1$ and $g \leq \psi_2$ on E.

$\psi_1 + \psi_2$ is simple and $f + g \leq \psi_1 + \psi_2$ on E. Thus

$$\int_E (f + g) \leq \int_E \psi_1 + \psi_2 = \int_E \psi_1 + \int_E \psi_2 .$$

So $\int_E (f+g) \leq \inf_{\psi_1 \geq f \ \psi_2 \geq g} (\int_E \psi_1 + \int_E \psi_2) = \int_E f + \int_E g.$

Similarly, if φ_1 and φ_2 are simple functions with $\varphi_1 \leq f$ and $\varphi_2 \leq g$ we get $\int_E (f+g) \geq \int_E f + \int_E g.$

Thus $\int_E (f+g) = \int_E f + \int_E g.$

To prove monotonicity assume $f \leq g$ on E and let $h = g - f \geq 0.$

By linearity: $\int_E g - \int_E f = \int_E (f-g) = \int_E h \geq 0\,.$

So $\int_E f \leq \int_E g.$

Corollary: Let f be a bounded measurable function on a set of finite measure E. Suppose A and B are disjoint measurable subsets of E. Then:

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof: $(f)(\chi_A)$ and $(f)(\chi_B)$ are bounded measurable functions on E and

$$f = (f)(\chi_A) + (f)(\chi_B).$$

For any bounded measurable subset $E_1 \subseteq E$

$$\int_{E_1} f = \int_E (f)(\chi_{E_1})\,.$$

Thus: $\int_{A\cup B} f = \int_E (f)(\chi_{A\cup B}) = \int_E [(f)(\chi_A) + (f)(\chi_B)]$

$$= \int_E (f)(\chi_A) + \int_E (f)(\chi_B) = \int_A f + \int_B f.$$

Corollary: Let f be a bounded measurable function on a set of finite measure E. Then $\left|\int_E f\right| \le \int_E |f|$.

Proof: $|f|$ is bounded and measurable, hence integrable. In addition:

$$-|f| \le f \le |f|.$$

Thus $-\int_E |f| \le \int_E f \le \int_E |f| \quad \Rightarrow \quad \left|\int_E f\right| \le \int_E |f|$.

Prop. Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E. If $f_n \to f$ uniformly on E, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof: Since $f_n \to f$ uniformly on E and each f_n is bounded, the limit f must be bounded.

f is measurable because it's the pointwise limit of a sequence of measurable functions (uniform convergence implies pointwise convergence).

Let $\epsilon > 0$. Choose N such that if $n \ge N$ then:

$$|f - f_n| < \frac{\epsilon}{m(E)} \quad \text{on } E.$$

By linearity and monotonicity:

$$\left|\int_E f - \int_E f_n\right| = \left|\int_E (f - f_n)\right| \le \int_E |f - f_n| \le \left(\frac{\epsilon}{m(E)}\right)\big(m(E)\big) = \epsilon.$$

Thus $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Ex. We saw in the example:

Let $f_n(x) = 0$ if $\frac{1}{n} < x \leq 1$ or $x = 0$

$= n$ if $0 < x \leq \frac{1}{n}$.

$\lim_{n\to\infty} \int_0^1 f_n = 1$, but $\int_0^1 f = 0$. So $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

The "problem" here is that $f_n \to f = 0$ pointwise, but not uniformly.

The Bounded Convergence Theorem: Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, i.e. there is a number $M \geq 0$ such that $|f_n| \leq M$ on E for all n. If $f_n \to f$ pointwise on E then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof: Since $f_n \to f$ pointwise on E and f_n is measurable for all n, f is measurable.

Since $|f_n| \leq M$ on E for all n, $|f| \leq M$.

So f is bounded and measurable on E, thus it's integrable over E.

By Egoroff's theorem we know that $f_n \to f$ uniformly on $B \subseteq E$, where $m(E \sim B)$ is "small".

We must show that for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|\int_E f_n - \int_E f| < \epsilon$.

Notice that:

$$\int_E f_n - \int_E f = \int_E (f_n - f) = \int_B (f_n - f) + \int_{(E\sim B)} (f_n - f)$$

$$= \int_B (f_n - f) + \int_{E\sim B} f_n + \int_{E\sim B} -f.$$

Thus we have:

$$|\int_E f_n - \int_E f| = |\int_B (f_n - f) + \int_{E\sim B} f_n + \int_{E\sim B} -f\,|$$

$$\leq \int_B |(f_n - f)| + \int_{E\sim B} |f_n| + \int_{E\sim B} |-f|$$

$$\leq \int_B |f_n - f| + 2M\big(m(E\sim B)\big).$$

Let $\epsilon > 0$. By Egoroff's theorem we can choose B so that $f_n \to f$ uniformly on B, and $m(E\sim B) < \frac{\epsilon}{4M}$.

Since $f_n \to f$ uniformly on B, there exists an N such that if $n \geq N$ then

$$|f_n - f| < \frac{\epsilon}{2(m(E))} \text{ on } B.$$

Thus we have:

$$|\int_E f_n - \int_E f| \leq \int_B |f_n - f| + 2M\big(m(E\sim B)\big)$$

$$\leq \frac{\epsilon}{2(m(E))}\big(m(B)\big) + 2M\left(\frac{\epsilon}{4M}\right) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus if $f_n \to f$ pointwise (but not uniformly) on E, and $\{f_n\}$ are uniformly bounded then we do have $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Ex. Let $f_n(x) = x^n \quad 0 \le x \le 1$. Then $f_n \to f$ pointwise where

$$f(x) = 0 \qquad \text{if } 0 \le x < 1$$
$$= 1 \qquad \text{if } x = 1.$$

$\{f_n\}$ does not converge uniformly to f, but $|f_n(x)| \le 1$ for all $0 \le x \le 1$, so $\{f_n\}$ is uniformly pointwise bounded on $[0,1]$.

Thus by the previous theorem $\lim\limits_{n\to\infty} \int_{[0,1]} f_n = \int_{[0,1]} f$.

In fact we can check this with the following calculation:

$$\int_0^1 x^n = \frac{x^{n+1}}{n+1}\Big|_{x=0}^{x=1} = \frac{1}{n+1}; \quad \text{Thus } \lim_{n\to\infty} \int_0^1 x^n = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

$\int_0^1 f = 0$ so $\lim\limits_{n\to\infty} \int_{[0,1]} f_n = \int_{[0,1]} f$.