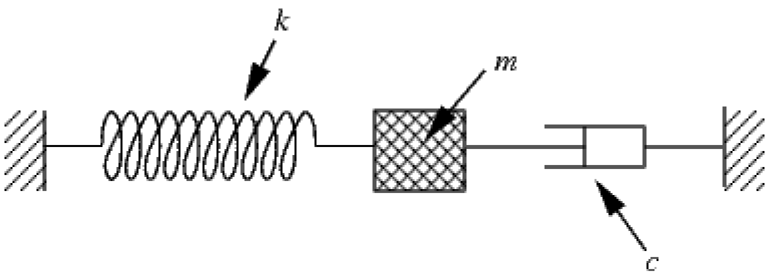
Vibrating Springs

Consider a body of mass, m, attached to one end of a spring while the other end of the spring is attached to the wall. Assume the body rests on a frictionless horizontal plane.

A dashpot is a device, like a shock absorber, that produces a force directed opposite to the direction of the mass. Assume the mass is attached to a dashpot (we can also think of this as representing any frictional force like air resistance).



$x(t) > 0$ if spring is stretched.

$x(t) < 0$ if spring is compressed.

The force from the spring is governed by Hooke's Law:

$$F_s = -kx, \quad k = \text{spring constant} > 0.$$

Assume the dashpot is designed so that:

$$F_R = -cv = -c\frac{dx}{dt}, \quad c > 0.$$

In the absence of external forces, we refer to the motion in this case as free, Newton's Law says:

$$F = ma = m\frac{d^2x}{dt^2} = mx'' \text{ and}$$

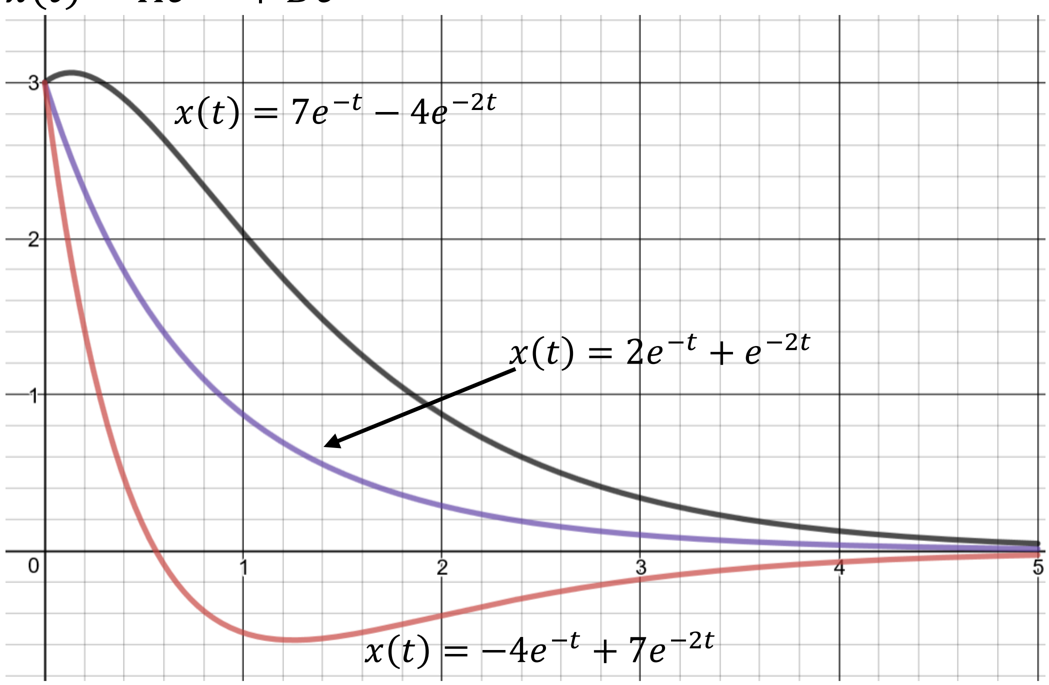
$$F = F_S + F_R = -kx - c\frac{dx}{dt} \text{ or}$$

$$mx'' + cx' + kx = 0.$$

There are three fundamentally different solutions to this differential equation based on the roots of the characteristic equation:

$$mr^2 + cr + k = 0$$

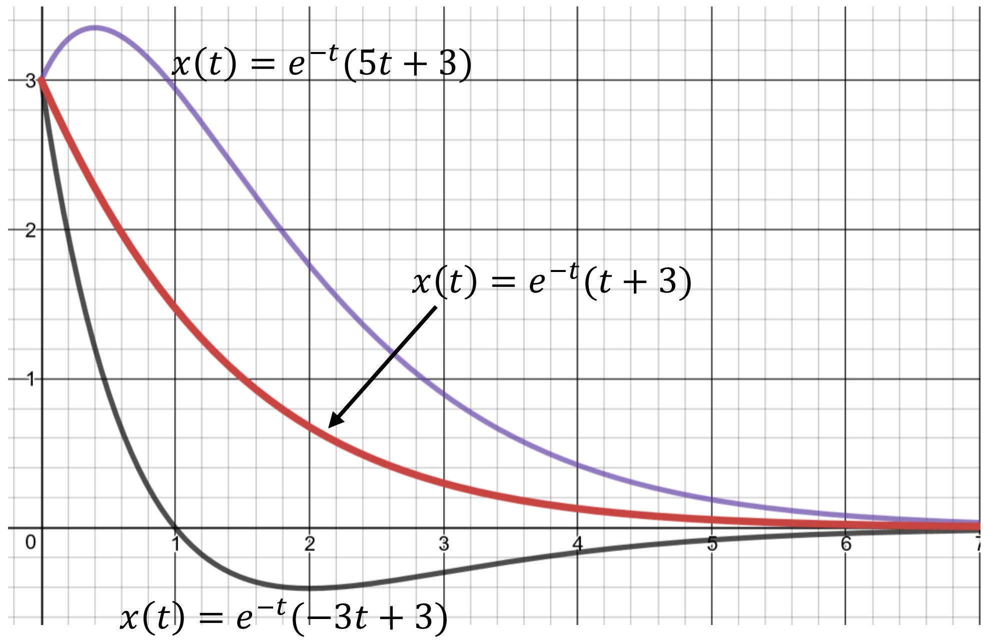
1) Overdamped case
   $c^2 - 4km > 0$;   2 real roots, both negative since $m, c, k > 0$
   $x(t) = Ae^{r_1 t} + Be^{r_2 t}$



2) Critically damped case

$c^2 - 4km = 0$; 2 equal negative roots

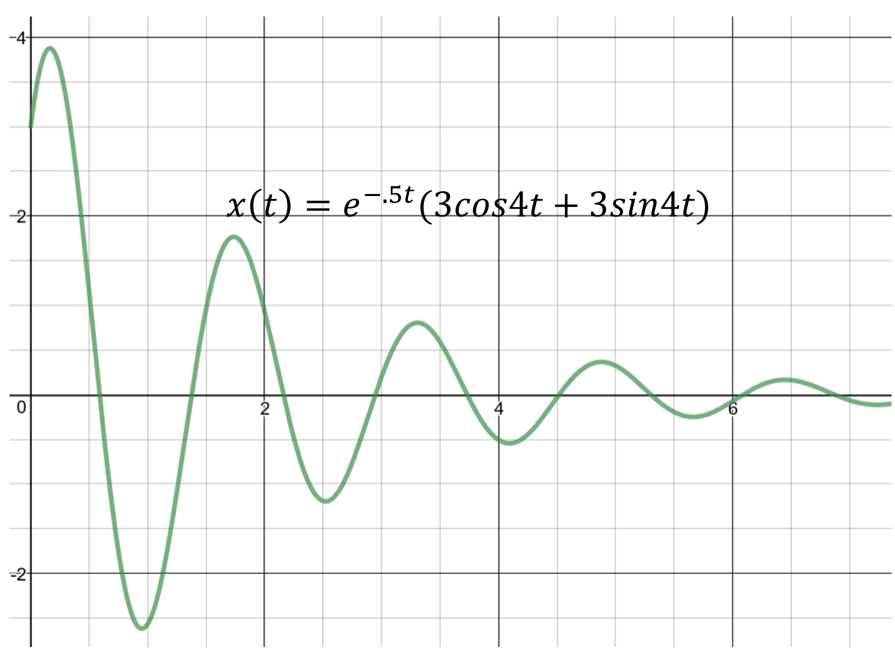
$x(t) = e^{rt}(A + Bt)$



3) Underdamped case

$c^2 - 4km < 0$; non-real, conjugate roots, $a \pm bi$ (a is negative)

$x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$.



If there is an external force, $F(t)$, put on the mass then the differential equation becomes:

$$mx'' + cx' + kx = F(t)$$

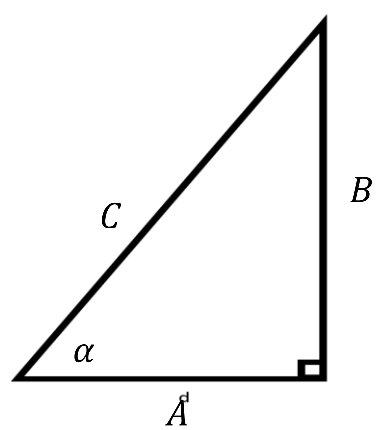
If $F(t) = 0$, we call the motion **Free**. If $F(t) \neq 0$, we call the motion **Forced**. For now, we will just consider a free system with no external force, so $F(t) = 0$.

Free Undamped Motion

If $c = 0$ we say we have **undamped** motion. So the differential equation generating the position $x(t)$ is $mx'' + kx = 0$, or we can write $x'' + \frac{k}{m}x = 0$.

If we define $w_0 = \sqrt{\frac{k}{m}}$ then we have $x'' + {w_0}^2 x = 0$ with the characteristic equation $r^2 + {w_0}^2 = 0$ or $r = \pm\, iw_0$. Thus, we can say that $x(t) = A\cos w_0 t + B\sin w_0 t$.

We can put the RHS in a more useful form by letting:



$C = \sqrt{A^2 + B^2}$, $\cos\alpha = \frac{A}{C}$, *and* $\sin\alpha = \frac{B}{C}$, so that $\tan\alpha = \frac{B}{A}$.

We have to be careful when we solve for α because the inverse tangent only has values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and A, B can be positive or negative. Thus, we get:

$$0 \leq \alpha < 2\pi; \quad \text{but} \quad -\frac{\pi}{2} < \tan^{-1} p < \frac{\pi}{2}, \quad p \in \mathbb{R}.$$

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) \qquad \text{If } A, B > 0 \ \ (1st \text{ quadrant})$$

$$= \pi + \tan^{-1}\left(\frac{B}{A}\right) \qquad \text{If } A < 0, \ (2nd \ \& \ 3rd \text{ quadrants})$$

$$= 2\pi + \tan^{-1}\left(\frac{B}{A}\right) \qquad \text{If } A > 0, B < 0 \ (4th \text{ quadrant}).$$

Now we can write:

$$x(t) = C\left(\frac{A}{C}\cos w_0 t + \frac{B}{C}\sin w_0 t\right)$$

$$= C(\cos\alpha\cos w_0 t + \sin\alpha\sin w_0 t)$$

$$= C(\cos(w_0 t - \alpha))$$

where Amplitude $= C$, Frequency $= \frac{w_0}{2\pi}$, Period $= \frac{2\pi}{w_0}$.

Free Damped Motion and Undamped Motion

Ex. Given the values of m, c, k, x_0, and v_0, find $x(t)$ and determine if the motion is overdamped, critically damped, or underdamped. If it's underdamped, write: $x(t) = Ce^{-pt}\cos(w_0 t - \alpha)$. Also find the undamped position $u(t) = C\cos(w_0 t - \alpha)$. That would result if the dashpot is disconnected (i.e. $c = 0$)

a) $m = \frac{1}{2}, \ c = 3, \ k = 4, \ x_0 = 2, \ v_0 = 0$
b) $m = 2, \ c = 12, \ k = 18, \ x_0 = 2, \ v_0 = -10$
c) $m = 1, \ c = 10, \ k = 125, \ x_0 = 6, \ v_0 = 50.$

a) $mx'' + cx' + kx = 0$

$c^2 - 4km = 3^2 - 4(4)\left(\frac{1}{2}\right) > 0$ so motion is overdamped.

$\frac{1}{2}x'' + 3x' + 4x = 0$

$x'' + 6x' + 8x = 0$

$r^2 + 6r + 8 = 0$

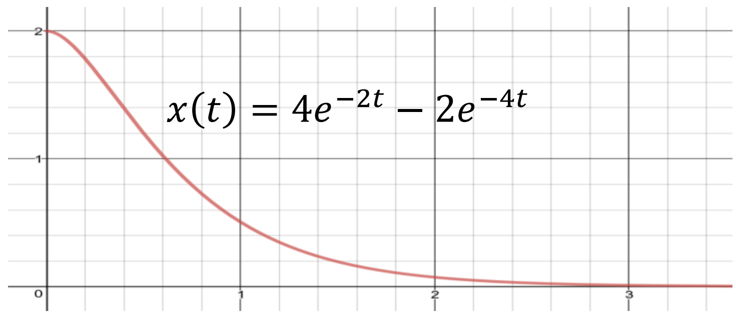
$(r+2)(r+4) = 0 \quad \Rightarrow \quad r = -2, -4$

$x(t) = Ae^{-2t} + Be^{-4t}; \qquad 2 = x(0) = A + B.$

$x'(t) = -2Ae^{-2t} - 4Be^{-4t}, \qquad 0 = x'(0) = -2A - 4B$

So $B = -2, \quad A = 4$ and

$x(t) = 4e^{-2t} - 2e^{-4t}$



Undamped case:

$c = 0$

$\frac{1}{2}u'' + 4u = 0$

$u'' + 8u = 0$

$r^2 + 8 = 0$

$r = \pm 2\sqrt{2}\, i$

$u(t) = A\cos\left(2\sqrt{2}t\right) + B(\sin 2\sqrt{2}\, t)$

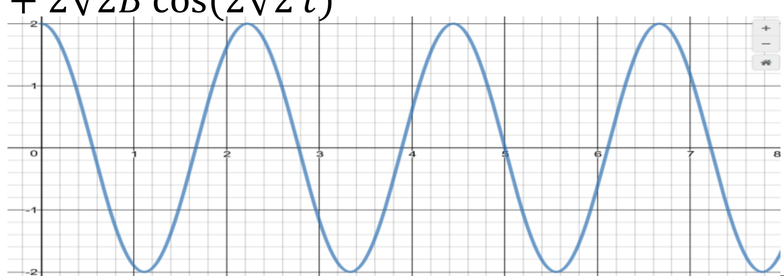
$u'(t) = -2\sqrt{2}A\sin\left(2\sqrt{2}t\right) + 2\sqrt{2}B\cos(2\sqrt{2}\, t)$

$2 = u(0) = A$

$0 = u'(0) = 2\sqrt{2}B$ so

$A = 2, \quad B = 0$ and

$u(t) = 2\cos(2\sqrt{2}\, t)$;



Amplitude $= 2$, Frequency $= \frac{2\sqrt{2}}{2\pi} = \frac{\sqrt{2}}{\pi}$, Period $= \frac{\pi}{\sqrt{2}}$.

b) $mx'' + cx' + kx = 0$; $\qquad c^2 - 4km = 12^2 - 4(18)(2) = 0$

$2x'' + 12x' + 18x = 0$ $\qquad$ so the motion is critically damped.

$$x'' + 6x' + 9x = 0$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)^2 = 0$$

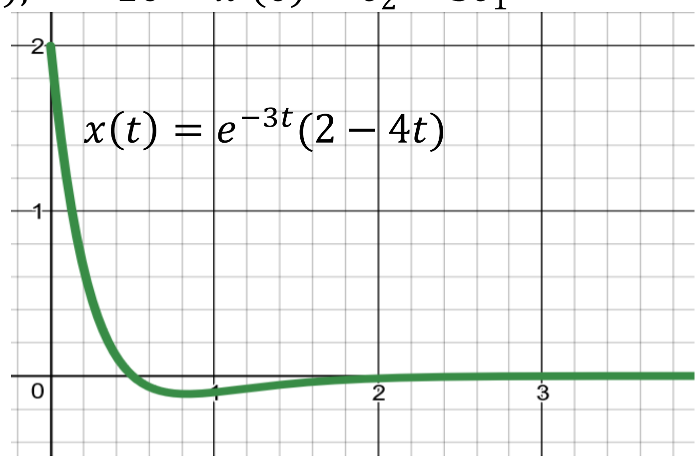
$r = -3$ double root

$x(t) = e^{-3t}(c_1 + c_2\, t)$; $\qquad 2 = x(0) = c_1$

$x'(t) = e^{-3t}(c_2) - 3e^{-3t}(c_1 + c_2\, t)$; $\qquad -10 = x'(0) = c_2 - 3c_1$

So $c_2 = -4$ and $c_1 = 2$

$x(t) = e^{-3t}(2 - 4t)$



Undamped case: $c = 0$

$$2u'' + 18u = 0$$

$$u'' + 9u = 0$$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

$u(t) = A\cos 3t + B\sin 3t$; $\qquad 2 = x(0) = A$

$u'(t) = -3A\sin 3t + 3B\cos 3t$; $\qquad -10 = x'(0) = 3B$

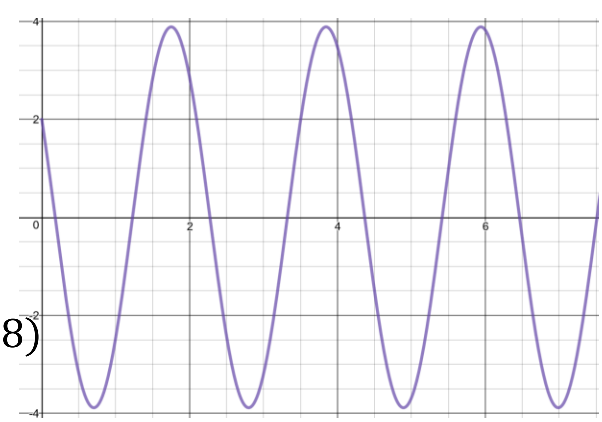
So $A = 2,\ \ B = -\frac{10}{3}$

$$C = \sqrt{2^2 + (\tfrac{10}{3})^2} = \sqrt{\tfrac{136}{9}} = \tfrac{2}{3}\sqrt{34}$$

$\alpha = \tan^{-1}\frac{\left(-\frac{10}{3}\right)}{2} + 2\pi$; $\quad$ since $B < 0,\ A > 0$

$\alpha = \tan^{-1}(-\frac{5}{3}) + 2\pi \approx 5.2528$ and

$u(t) = \frac{2}{3}\sqrt{34}\cos(3\,t - 5.2528)$.



$u(t) = \frac{2}{3}\sqrt{34}\cos(3\,t - 5.2528)$

$mx'' + cx' + kx = 0\,;\qquad c^2 - 4km = 10^2 - 4(1)(125) < 0$

$x'' + 10x' + 125x = 0$ so the motion is underdamped.

$r^2 + 10r + 125 = 0$

$$r = \frac{-10 \pm \sqrt{100-500}}{2} = -5 \pm 10i$$

$x(t) = e^{-5t}(A\cos 10t + B\sin 10t)$

$x'(t) = e^{-5t}(-10A\sin 10t + 10B\cos 10t)$
$\qquad -5e^{-5t}(A\cos 10t + B\sin 10t)$

$6 = x(0) = A$

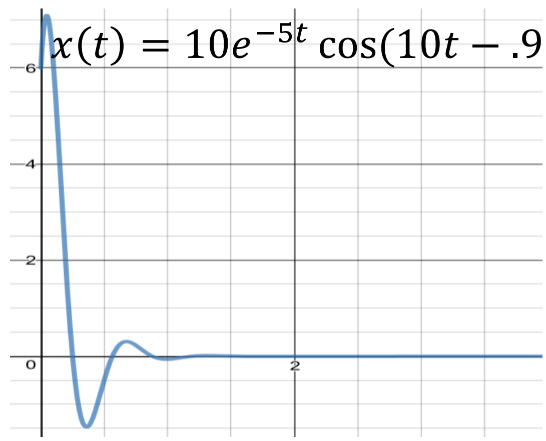
$50 = x'(0) = 1(0 + 10B) - 5(1)(A + 0) = 10B - 5A$

$A = 6, \qquad B = 8$

$C = \sqrt{6^2 + 8^2} = 10$

$\alpha = \tan^{-1}(\frac{8}{6}) \approx .9273\,,$ since $A, B > 0$.

$x(t) = 10e^{-5t}\cos(10t - .9273)$



Undamped case: $c = 0$

$u'' + 125u = 0$

$r^2 + 125 = 0$

$r = \pm 5\sqrt{5}i$

$u(t) = A\cos(5\sqrt{5}\,t) + B\sin(5\sqrt{5}\,t)$

$u'(t) = -5\sqrt{5}A\sin(5\sqrt{5}\,t) + 5\sqrt{5}B\cos(5\sqrt{5}\,t)$

$6 = u(0) = A$

$50 = u'(0) = 5\sqrt{5}B \;\Rightarrow\; B = \frac{50}{5\sqrt{5}} = \frac{10}{\sqrt{5}}\,.$

$C = \sqrt{A^2 + B^2} = \sqrt{6^2 + \frac{100}{5}} = \sqrt{56} = 2\sqrt{14}$

$\alpha = \tan^{-1}(\frac{\frac{10}{\sqrt{5}}}{6}) = \tan^{-1}(\frac{10}{6\sqrt{5}}) \approx .6405\,,$ since $A, B > 0$.

$u(t) = 2\sqrt{14}\cos\big(5\sqrt{5}t - .6405\big).$