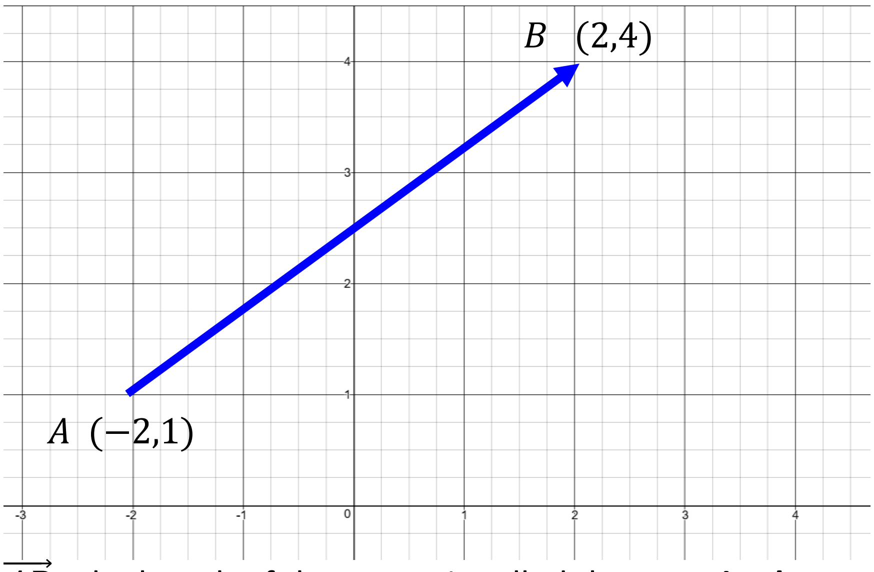
# Vectors in the Plane and Three-Space

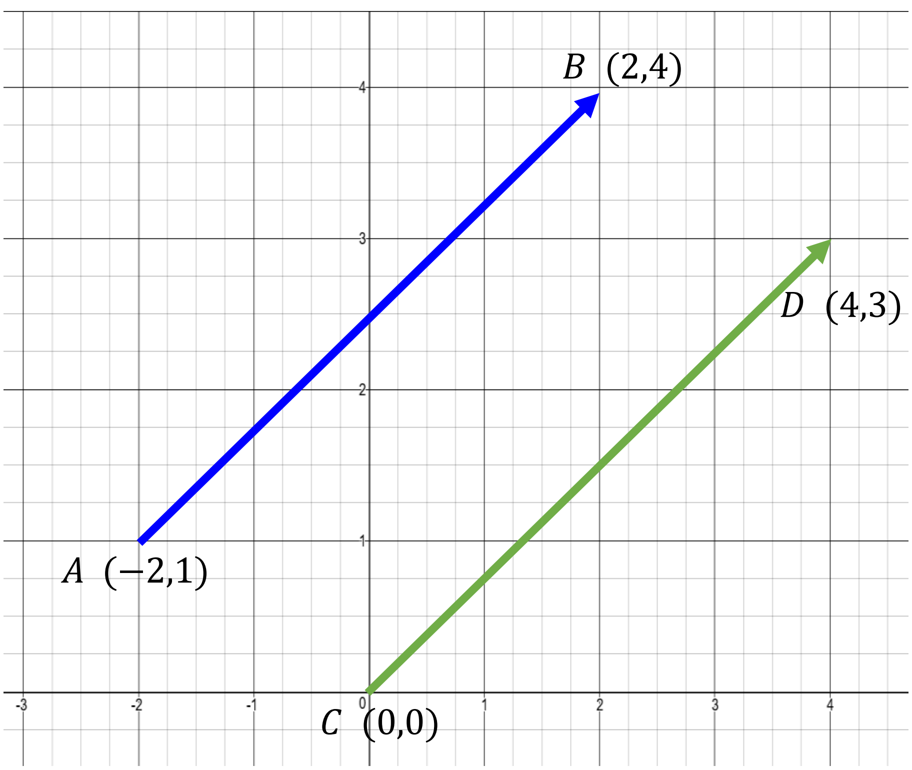
A **vector** is a quantity that has magnitude and direction, for example, velocity or force. It is often represented by an arrow.

A vector has an initial point (the tail) and a terminal point (the tip).




$\overrightarrow{AB}$ ; the length of the vector is called the **magnitude**.

Since all that matters with a vector is magnitude and direction, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal (or the same).

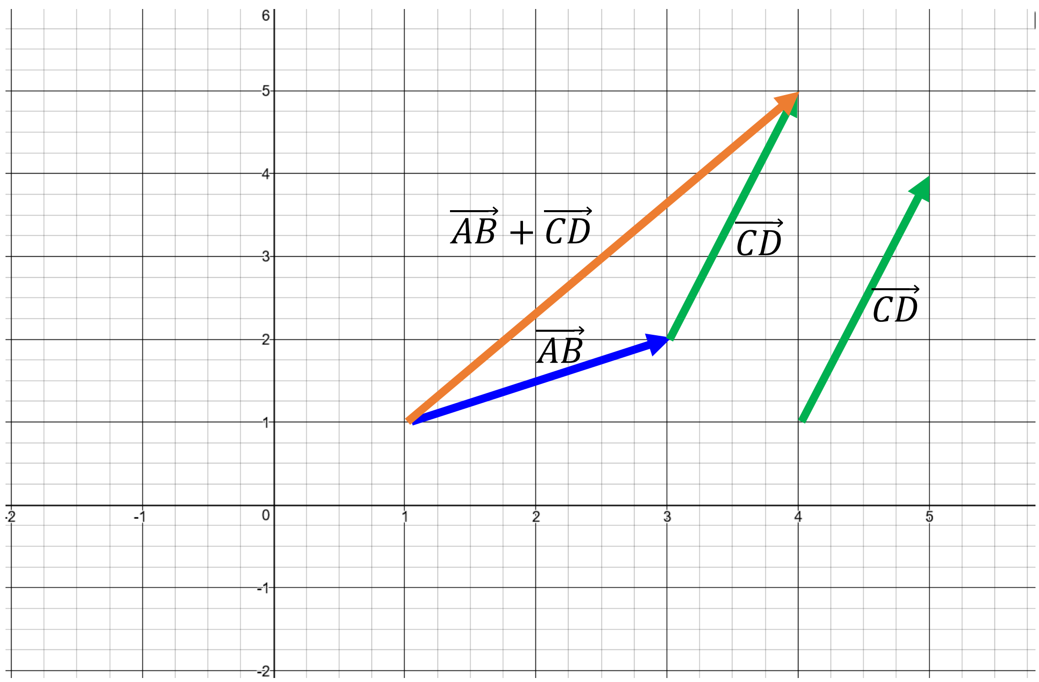


$\overrightarrow{AB} = \overrightarrow{CD}$, so we can "move' vectors around as long as we don't change the magnitude (length) or direction (can move parallel).

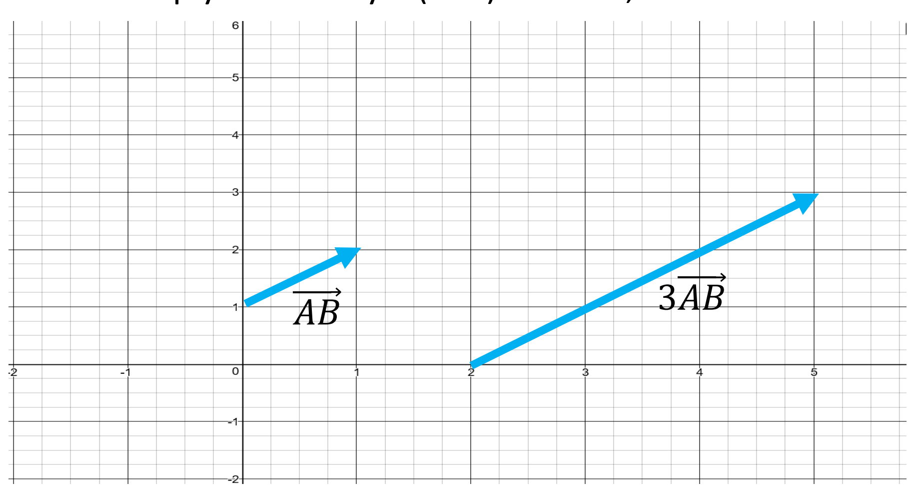
Given 2 points in either $\mathbb{R}^2$ or $\mathbb{R}^3$, say $A(-2,1)$ and $B(2,4)$, we can create a vector by subtracting the coordinates:

$$\overrightarrow{AB} = (2-(-2), 4-1) = < 4,3 >.$$

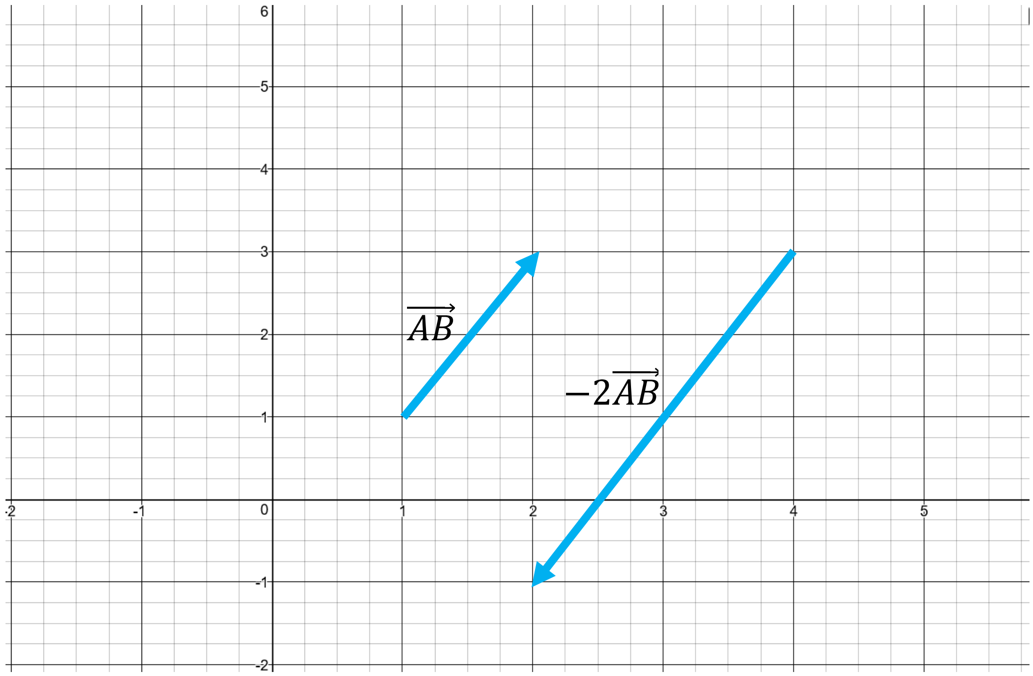
To add vectors we put the tail of one to the tip of the other:



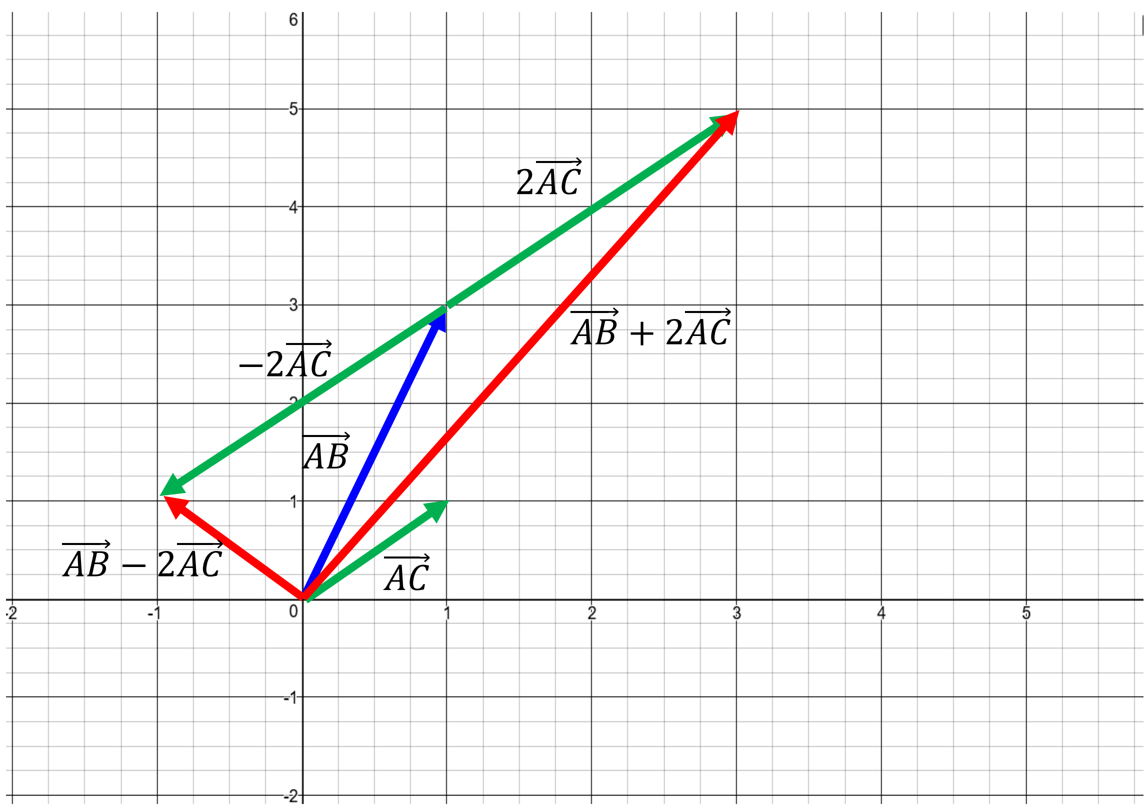

We can multiply vectors by a (real) number, called a **scalar**:



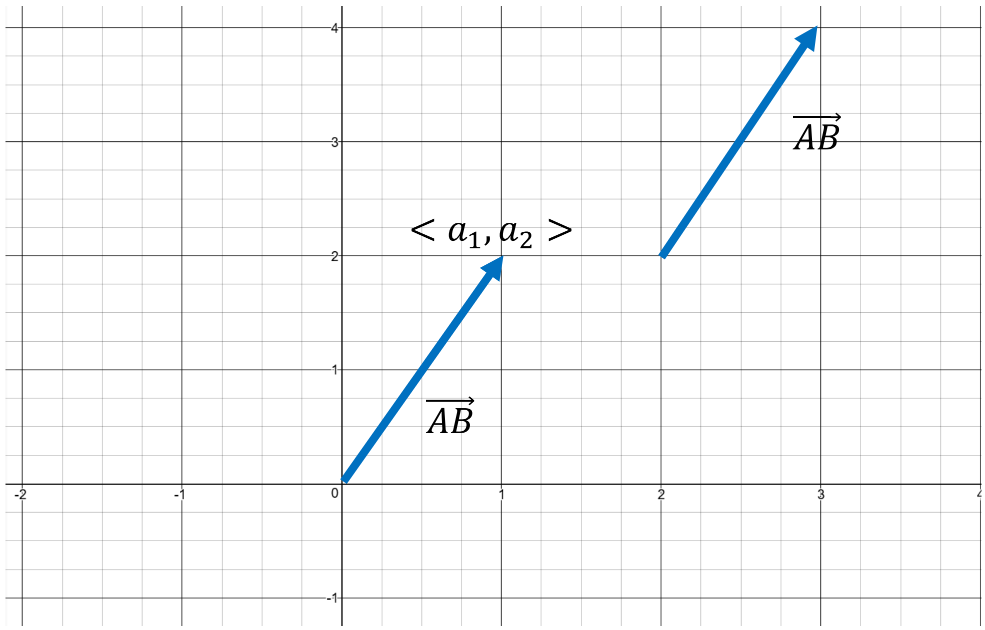
If we multiply a vector by a negative number it creates a vector in the opposite direction:




Ex. Find $\overrightarrow{AB} + 2\overrightarrow{AC}$ and $\overrightarrow{AB} - 2\overrightarrow{AC}$:



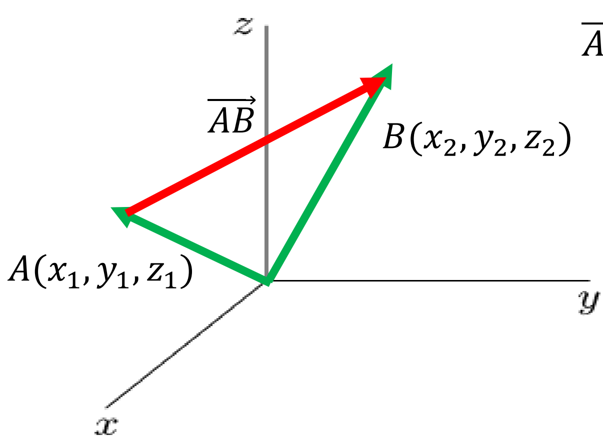
Since all that matters for a vector are its magnitude and direction, we can think of any vector as having its tail at the origin and the tip at a point (this can be done for any number of dimensions).




The $a_i$s are called the **components** of the vector.

When we mean a vector, we will write $< a_1, a_2, a_3 >$, as opposed to the point $(a_1, a_2, a_3)$.

Given the points: $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector:




$$\overrightarrow{AB} = < x_2 - x_1,\ \ y_2 - y_1,\ \ z_2 - z_1 >$$

Notice the order of subtraction matters: $\overrightarrow{AB} \Rightarrow A$ is tail, $B$ is tip

Ex. Find the vector from $A(-2,3,-1)$ (tail) to $B(-1,5,2)$ (tip).

$$\overrightarrow{AB} = < -1-(-2), 5-3, 2-(-1) > = < 1,2,3 >$$

The magnitude, or length, of a vector is given by the distance formula:

In $\mathbb{R}^2$: $\vec{u} = < u_1, u_2 >$ $\quad |\vec{u}|$ or $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$

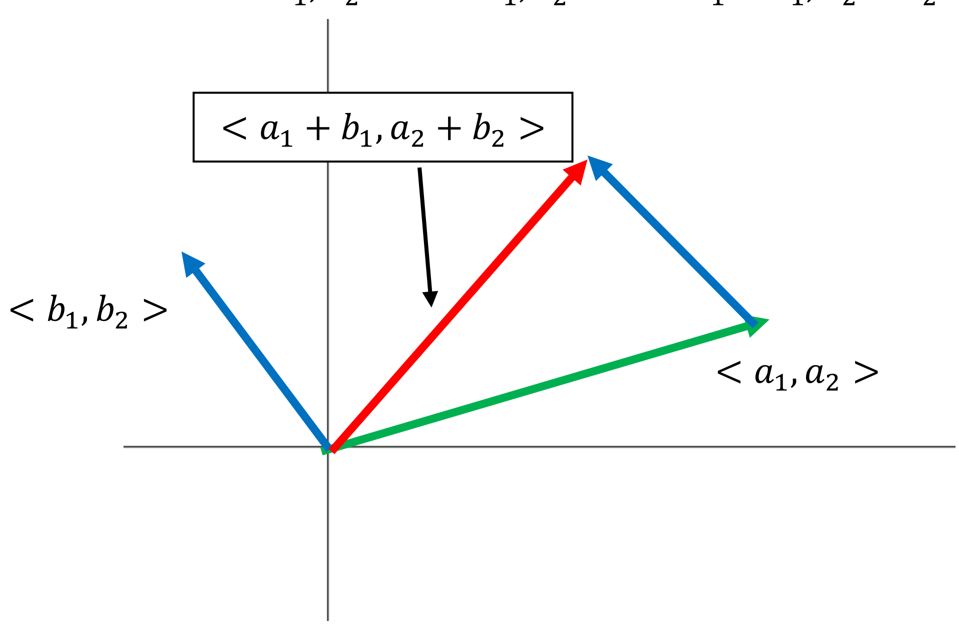
In $\mathbb{R}^3$: $\vec{u} = < u_1, u_2, u_3 >$ $\quad |\vec{u}|$ or $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Ex. Find the length of $\vec{u} = < -1, 4, -2 >$.

$$|\vec{u}| = \sqrt{(-1)^2 + 4^2 + (-2)^2} = \sqrt{1+16+4} = \sqrt{21}.$$

To add (or subtract) vectors we add (or subtract) their components.

$$< a_1, a_2 > + < b_1, b_2 > = < a_1 + b_1, a_2 + b_2 >$$



Ex. $\vec{A} = < 2, -3, -1 >$ , $\vec{B} = < -3, 2, 4 >$.

Find $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$.

$$\vec{A} + \vec{B} = < 2 - 3,\ -3 + 2,\ -1 + 4 > = < -1,\ -1,\ 3 >$$

$$\vec{A} - \vec{B} = < 2 - (-3),\ -3 - 2,\ -1 - 4 > = < 5,\ -5,\ -5 >$$

To multiply a vector by a scalar, multiply the components.

Ex. If $c = -3$, $\vec{A} = < 1, -2, 3 >$

then $c\vec{A} = < -3(1),\ -3(-2),\ -3(3) > = < -3, 6, -9 >$

Ex. Let $\vec{A} = < -3, 0, 5 >$ , $\vec{B} = < -2, -1, 3 >$. Find $\vec{A} - 2\vec{B}$ and $|\vec{A} - 2\vec{B}|$.

$$\begin{aligned}\vec{A} - 2\vec{B} &= < -3, 0, 5 > -2 < -2, -1, 3 > \\ &= < -3, 0, 5 > - < -4, -2, 6 > \\ &= < -3 - (-4),\ 0 - (-2),\ 5 - 6 > = < 1, 2, -1 >.\end{aligned}$$

$$|\vec{A} - 2\vec{B}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

Properties of vectors: If $\vec{A}, \vec{B}$, and $\vec{C}$ are vectors and $c, d \in \mathbb{R}$ then

1. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
2. $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$
3. $\vec{A} + \vec{0} = \vec{A}$
4. $\vec{A} + (-\vec{A}) = \vec{0}$
5. $1\vec{A} = \vec{A}$
6. $c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$
7. $(c + d)\vec{A} = c\vec{A} + d\vec{A}$
8. $(cd)\vec{A} = c(d\vec{A})$

Proof of Property 1:

Let $\vec{A} = < a_1, a_2, a_3 >$ , $\vec{B} = < b_1, b_2, b_3 >$

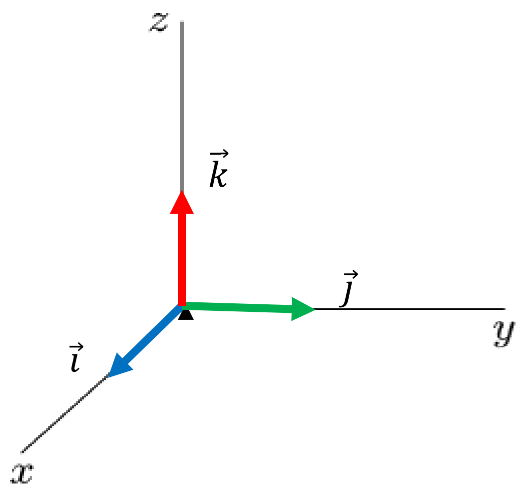
$$\vec{A} + \vec{B} = < a_1 + b_1, a_2 + b_2, a_3 + b_3 > = < b_1 + a_1, b_2 + a_2, b_3 + a_3 > = \vec{B} + \vec{A}$$

Three vectors play a special role and are given the names:

$$\vec{\imath} = < 1, 0, 0 > , \quad \vec{\jmath} = < 0, 1, 0 > , \quad \vec{k} = < 0, 0, 1 >$$

These 3 vectors are called the standard basis for $\mathbb{R}^3$.

They are of length 1 and perpendicular.




For $\mathbb{R}^2$, $\vec{\imath} =< 1, 0 >$ , $\vec{\jmath} = < 0, 1 >$.

We can write any vector in $\mathbb{R}^3$ in terms of these vectors:

For example: $< 5, -2, 3 > = 5\vec{\imath} - 2\vec{\jmath} + 3\vec{k}$

In general, $< a_1, a_2, a_3 > = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$.

Ex. Let $\vec{A} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$, $\vec{B} = -3\vec{\imath} + 6\vec{k}$. Find $\vec{A} + \vec{B}$.

$$\vec{A} + \vec{B} = \left(2\vec{\imath} - \vec{\jmath} + 3\vec{k}\right) + (-3\vec{\imath} + 6\vec{k})$$
$$= -\vec{\imath} - \vec{\jmath} + 9\vec{k}$$

A unit vector is a vector of length 1. Given any vector, $\vec{A} \neq \vec{0}$, we can create a unit vector in the same direction, $\vec{u}$, by

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

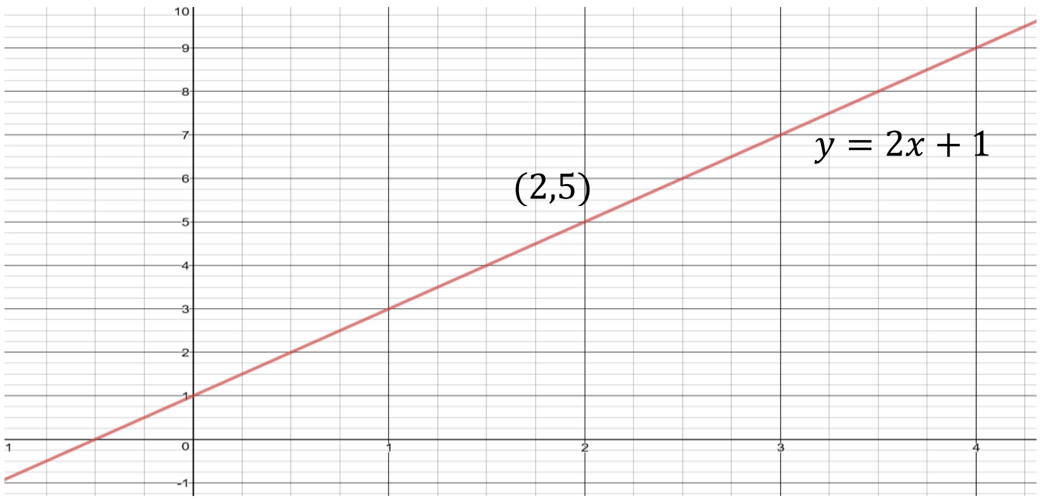
Ex. Find a unit vector in the direction of $\vec{A} = < -2, 3, 2 > = -2\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$.

$$|\vec{A}| = \sqrt{(-2)^2 + 3^2 + 2^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$$
$$\vec{u} = \frac{\vec{A}}{\sqrt{17}} = < \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}} > = -\frac{2}{\sqrt{17}}\vec{\imath} + \frac{3}{\sqrt{17}}\vec{\jmath} + \frac{2}{\sqrt{17}}\vec{k}\ .$$

Equations of Lines in $\mathbb{R}^2$

If we know a point on a line and the direction (i.e. slope), we can write an equation of a line. For example, if the point $(2,5)$ is on the line and the slope of the line is 2, we know an equation of the line is:

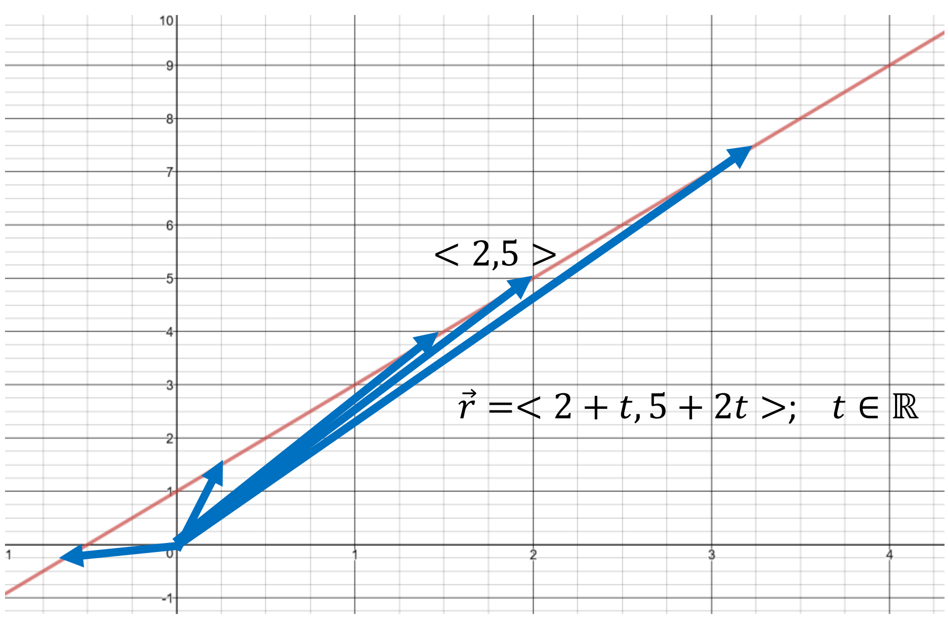
$$y - 5 = 2(x - 2) \quad \text{or} \quad y = 2x + 1.$$




If we want to write this in vector form we can take any point on the line, say $(2,5)$, and consider the vector $< 2,5 >$. Notice that the vector $< 1,2 >$ has a slope of 2. We can think of the line as the tips of a set of vectors given by:

$$\vec{r} = < 2,5 > + t < 1,2 > = < 2 + t, 5 + 2t >; \quad t \in \mathbb{R}$$

This is the vector form of a line in $\mathbb{R}^2$.

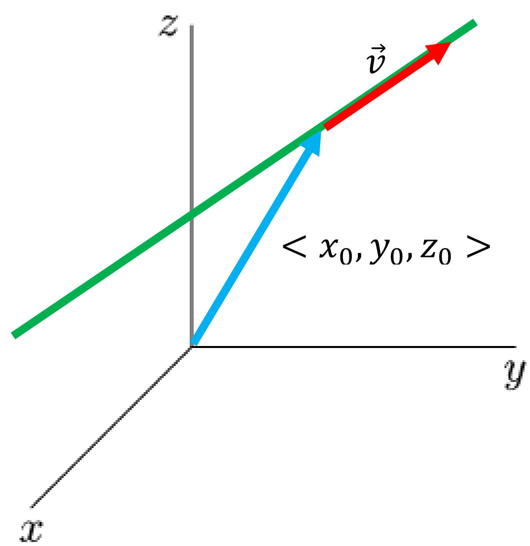


Equations of Lines in $\mathbb{R}^3$

The vector form of a line.

Take any point on the line and write it as a vector $< x_0, y_0, z_0 >$ and then find a direction vector $\vec{v} =< v_1, v_2, v_3 >$. The vector equation of the line is given by:

$$\vec{r} = < x_0, y_0, z_0 > \ + t < v_1, v_2, v_3 >, \quad \text{where } t \in \mathbb{R}.$$




Ex. Write the vector form of a line going through $(2, -3, 1)$ and parallel to the vector $< 4, 3, -5 >$.

$$\vec{r} = < 2, -3, 1 > + t < 4, 3, -5 >; \qquad t \in \mathbb{R}.$$

Parametric equation form of a line

We can start with the vector form of a line as:

$$\vec{r} = \vec{r}_0 + t\vec{v} \qquad t \in \mathbb{R}$$

$$< x, y, z > = < x_0, y_0, z_0 > +t < v_1, v_2, v_3 >$$

$$= < x_0 + tv_1,\ y_0 + tv_2\ ,\ z_0 + v_3 >.$$

$$x = x_0 + tv_1$$

$$y = y_0 + tv_2 \qquad t \in \mathbb{R}$$

$$z = z_0 + tv_3$$

is called the **parametric equation form of a line**.

Ex. Find the parametric equations of a line through $(2, -3, 1)$ in the direction of (or parallel to) $< 4, 3, -5 >$.

$$\vec{r} = < 2, -3, 1 > + t < 4, 3, -5 >$$

$$x = 2 + 4t$$

$$y = -3 + 3t \qquad t \in \mathbb{R}$$

$$z = 1 - 5t.$$

If $\vec{v} = < a, b, c >$ is used to describe the direction of a line, $L$, then $a, b, c$ are called **direction numbers of $L$**. Of course, any non-zero multiple of $\vec{v}$ is parallel to $\vec{v}$, so any non-zero multiples of $a, b, c$ would also be direction numbers for $L$.

Ex. Find parametric equations of the line that goes through $A(-6,-1,2)$, $B(-3,2,1)$. At what point does the line intersect the $yz$ plane?

Direction vector $= \overrightarrow{AB} = < -3 - (-6),\ 2 - (-1),\ 1 - 2 >$

$= < 3, 3, -1 >.$

The line goes through $(-6,-1,2)$ (we could use either point).

The vector form of the line is:

$$\vec{r} = < -6, -1, 2 > + t < 3, 3, -1 > \qquad t \in \mathbb{R}.$$

Thus the parametric equation form of the line is:

$$x = -6 + 3t$$

$$y = -1 + 3t$$

$$z = 2 - t.$$

The line intersects the $yz$ plane when $x = 0$. When $x = 0, t = 2$.

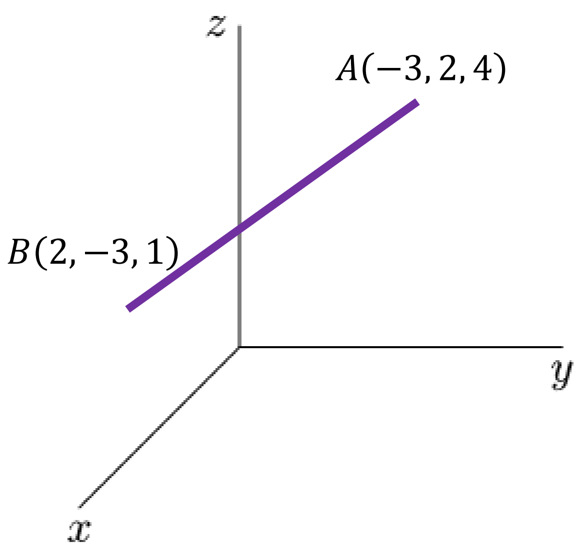
Thus the line intersects the $yz$ plane at: $t = 2$, so at

$$x = -6 + 3(2) = 0, \quad y = -1 + 3(2) = 5, \quad z = 2 - 2 = 0,$$

$$(x, y, z) = (0,5,0).$$

Sometimes we need the description of a line segment instead of an entire line.

Ex. Find a vector form of a line segment starting at $A(-3, 2, 4)$ and ending at $B(2, -3, 1)$.




$$\overrightarrow{AB} = <2-(-3), -3-2, 1-4> = <5, -5, -3>$$

Equation of the line:

$$\vec{r} = <-3, 2, 4> + t <5, -5, -3>; \quad t \in \mathbb{R}$$

Notice: $t = 0, \quad \vec{r} = <-3, 2, 4>$

$t = 1, \quad \vec{r} = <2, -3, 1>$

Line segment in vector form:

$$\vec{r}(t) = <-3, 2, 4> + t <5, -5, -3>; \quad 0 \le t \le 1.$$

Notice that in the previous example:

$\overrightarrow{AB} = < 5, -5, -3 > = \vec{B} - \vec{A} = < 2, -3, 1 > - < -3, 2, 4 >$

So the line segment is:

$$\vec{r}(t) = \vec{A} + t\left(\vec{B} - \vec{A}\right) = \vec{A} - t\vec{A} + t\vec{B} = (1 - t)\vec{A} + t\vec{B}$$

$$\vec{r}(t) = (1 - t)\vec{A} + t\vec{B}\,; \qquad 0 \le t \le 1$$

where $\vec{A} = \vec{r}(0)$ is the vector with tip at the starting point $A$ and

$\vec{B} = \vec{r}(1)$ is the vector with tip at the ending point $B$.

Ex. Write a vector equation and parametric equations for the line segment starting at $A(3, -1, 5)$ and ending at $B(-2, 4, 2)$.

Vector equation:

$$\vec{r} = (1 - t) < 3, -1, 5 > +t < -2, 4, 2 >; \qquad 0 \le t \le 1$$

$$=< 3 - 3t,\ -1 + t,\ 5 - 5t > + < -2t,\ 4t,\ 2t >$$

$$= < 3 - 5t,\ -1 + 5t,\ 5 - 3t >; \qquad 0 \le t \le 1.$$

Parametric equations:

$$x = (1 - t)3 - 2t = 3 - 5t$$

$$y = (1 - t)(-1) + 4t = -1 + 5t \qquad 0 \le t \le 1$$

$$z = (1 - t)5 + 2t = 5 - 3t$$

Given 2 lines in $\mathbb{R}^3$, they can:

1) Intersect at 1 point

2) Be parallel (direction vectors are multiples, but lines don't intersect)

3) Be skew (don't intersect but are not parallel)

4) Intersect at every point (i.e. they're the same line)

Ex. Determine if the lines, $L_1$ and $L_2$, are parallel, skew, intersect at one point, or are the same line. If they intersect, find the point of intersection.

$L_1$: $x = 3t$ $\quad$ $L_2$: $x = 1 + 4s$

$y = 2 - t$ $\quad$ $y = -2 + s$

$z = -1 + t$ $\quad$ $z = -3 - 3s.$

$L_1$ direction vector: $< 3, -1, 1 >$ $\quad$ $L_2$ direction vector: $< 4, 1, -3 >$.

These are not multiples of each other so $L_1$ and $L_2$ are not parallel or the same line.

If they intersect, we could find numbers, $t, s$,

such that:

$$\begin{array}{lclcl} 3t = 1 + 4s & \Rightarrow & 3t - 4s = 1 & \Rightarrow & 3t - 4s = 1 \\ 2 - t = -2 + s & \Rightarrow & -t - s = -4 & \Rightarrow & \underline{-3t - 3s = -12} \\ -1 + t = -3 - 3s & & & & -7s = -11 \\ & & & & s = \frac{11}{7} \end{array}$$

$$\Rightarrow \qquad -t - \frac{11}{7} = -4$$

$$-t = -\frac{28}{7} + \frac{11}{7} = -\frac{17}{7}$$

$$t = \frac{17}{7}.$$

Now check if $s = \frac{11}{7},\ t = \frac{17}{7}$ fits the 3rd equation:

$$-1 + \frac{17}{7} = -3 - 3(\frac{11}{7})$$

$$\frac{10}{7} = -3 - \frac{33}{7}$$ doesn’t work, so the lines are skew.