Review of Analysis 1: Important Concepts and Definitions

The Triangle inequalities for $\mathbb{R}$ and $\mathbb{R}^n$:

a. If $x, y \in \mathbb{R}$ then $|x + y| \leq |x| + |y|$

b. If $\vec{v}, \vec{w} \in \mathbb{R}^n$ then $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$;

where $\vec{v} =< v_1, v_2, \dots, v_n >$ and $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$.

Def. A set X, whose elements are called points, is a **Metric Space** if for any two points $p, q \epsilon X$ there is a real number $d(p, q)$, called the distance (or metric), such that:

a. $d(p, q) > 0$ if $p \neq q$, and $d(p, p) = 0$.

b. $d(p, q) = d(q, p)$

c. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \epsilon X$ (Triangle inequality).

Ex. $X = \mathbb{R}$, $d(p, q) = |p - q|$ (the standard distance function on $\mathbb{R}$)

To show X, d is a metric space, we need to show that d satisfies a,b,c above.

a. $d(p, q) = |p - q| > 0$ if $p \neq q$ (property of absolute values),

$d(p, p) = |p - p| = 0$

b. $d(p, q) = |p - q| = |q - p| = d(q, p)$

c. To show $d(p, q) \leq d(p, r) + d(r, q)$, $r \epsilon X$, for this distance function means:

$|p - q| \leq |p - r| + |r - q|$

In the triangle inequality: $|x + y| \leq |x| + |y|$, let $x = p - r$ and $y = r - q$

Then we get: $|p - q| \leq |p - r| + |r - q|$.

Ex. Show $\mathbb{R}, d$ is a metric space where $d(p,q) = |e^p - e^q|$.

a. $d(p,q) = |e^p - e^q| > 0$ for $p \neq q$ because $f(x) = e^x$ is an increasing function, and $d(p,p) = |e^p - e^p| = 0$.

b. $d(p,q) = |e^p - e^q| = |e^q - e^p| = d(q,p)$.

c. We need to show: $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \epsilon \mathbb{R}$. In this case:

$$|e^p - e^q| \leq |e^p - e^r| + |e^r - e^q|.$$

This looks daunting, but remember the Triangle inequality for real numbers:

$$|x + y| \leq |x| + |y| .$$

Now let $x = e^p - e^r$ and $y = e^r - e^q$, so $x + y = e^p - e^q$. Hence:

$$|e^p - e^q| \leq |e^p - e^r| + |e^r - e^q|.$$

Hence $\mathbb{R}, d$ is a metric space.

Note: Not all metric spaces are subsets of $\mathbb{R}^n$.

Ex. $X = C[0,1]$ =set of real valued, continuous functions on [0,1]. X is a metric space with either of these 2 metrics (there are an infinite number of metrics on X)

$$d_1(f,g) = \int_0^1 |f(x) - g(x)| dx$$

$$d_2(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

Ex. Let $f(x) = x^2$ and $g(x) = x^3$. Notice that $f(x), g(x) \in C[0,1]$. Using the 2 metrics just defined on $C[0,1]$, find $d_1(f,g)$ and $d_2(f,g)$.

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx = \int_0^1 |x^2 - x^3|dx \,.$$

Notice that when $0 \leq x \leq 1,\ x^2 \geq x^3$ (when $0 < x < 1$ the higher the power the lower the value).

Thus when $0 \leq x \leq 1,\ x^2 - x^3 \geq 0$ hence $|x^2 - x^3| = x^2 - x^3$. So

$$d_1(f,g) = \int_0^1 |x^2 - x^3|dx = \int_0^1 (x^2 - x^3)dx$$

$$= \frac{1}{3}x^3 - \frac{1}{4}x^4\Big|_{x=0}^{x=1} = \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}\,.$$

$$d_2(f,g) = \max_{x\in[0,1]} |f(x) - g(x)| = \max_{x\in[0,1]} |x^2 - x^3|$$

To find the maximum value of $|h(x)|$, we need to find where $h(x)$ has its greatest positive value and its most negative value and choose the one which is greater in absolute value (e.g. if $h(x)$ has 4 as its most positive value and -6 as its most negative value then the maximum of $|h(x)|$ is |-6|=6.)

In this case we already saw that $x^2 - x^3 \geq 0$ so we just have to find the absolute maximum value of $h(x) = x^2 - x^3$. Using first year calculus, find the values of $h(x)$ at all critical points in [0,1] and then test the values at the endpoints.

$$h'(x) = 2x - 3x^2 = x(2 - 3x) = 0$$

$$\Rightarrow \qquad x = 0 \text{ or } x = \frac{2}{3}\,.$$

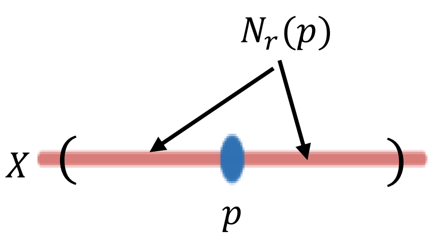
$$h(0) = 0\,, \quad h\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 = \frac{4}{9} - \frac{8}{27} = \frac{4}{27}\,, \quad h(1) = 1^2 - 1^3 = 0.$$

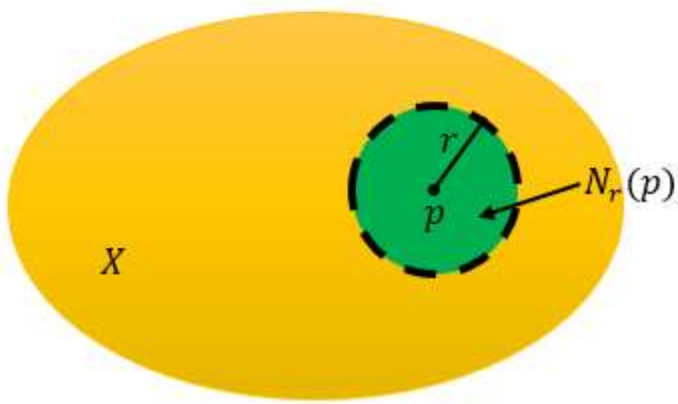
So the absolute maximum value of $h(x)$ is $\frac{4}{27}$ (absolute minimum is 0). So

$$d_2(f,g) = \max_{x\in[0,1]} |f(x) - g(x)| = \max_{x\in[0,1]} |x^2 - x^3| = \frac{4}{27}.$$

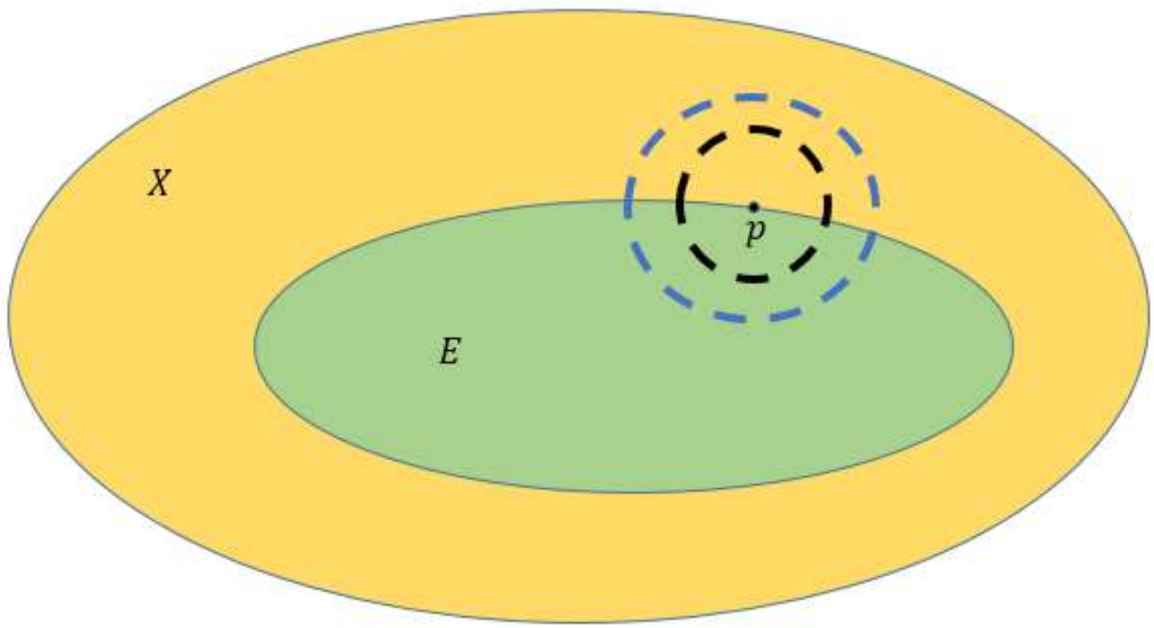
Def. Let X be a metric space with distance function d.

a. A **neighborhood of $\boldsymbol{p}$**, where $p\epsilon X$, is a set $N_r(p)$ of all points q such that $d(p,q) < r$ for some $r > 0$.



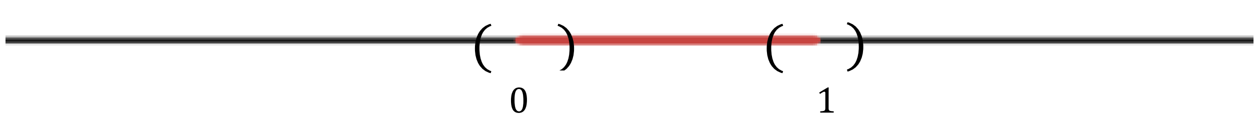


b. A point p is a **limit point** of $E \subseteq X$ if every neighborhood of p contains a point $q \neq p$ such that $q \epsilon E$.



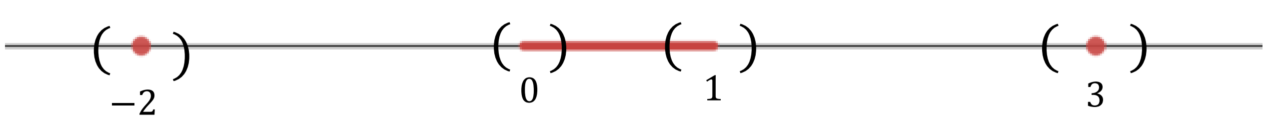
Ex. Let $X = \mathbb{R}$, and d the standard metric (i.e. $d(p,q) = |p - q|$). Let $E = (0,1)$

That is $E = \{x \epsilon \mathbb{R} |\ 0 < x < 1\}$. The set of limit points of $E = [0,1]$.



Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = (0,1) \cup \{3\} \cup \{-2\}$.

The set of limit points of $E = [0,1]$.



c. If $p \epsilon E$ and p is not a limit point of E, then p is called an **isolated point** of E.

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = (0,1) \cup \{3\} \cup \{-2\}$.

$\{3\}, \{-2\}$ are isolated points of E.

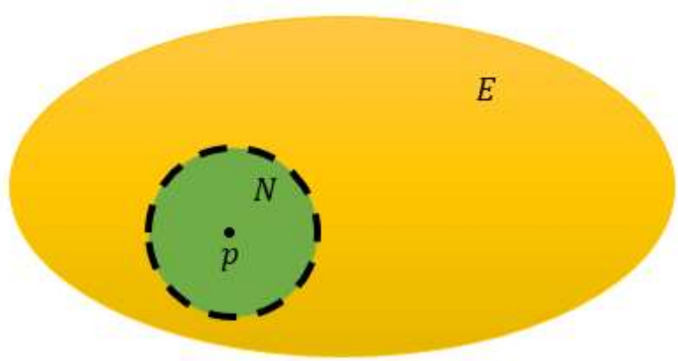
d. E is **closed** if every limit point of E is a point in E.

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1] \cup \{5\} \cup \{-3\}$.

E is closed in $X = \mathbb{R}$.

Let $F = (0,1] \cup \{5\} \cup \{-3\}$. F is not closed in $X = \mathbb{R}$, since $x = 0$ is a limit point of F, but is not contained in F.

e. A point p is an **interior point** of E if there is some neighborhood N of p, such that $N \subseteq E$.



Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1) \cup \{3\}$.

$0 < x < 1$ are interior points of E. $x = 0$ and $x = 3$ are not interior points of E.

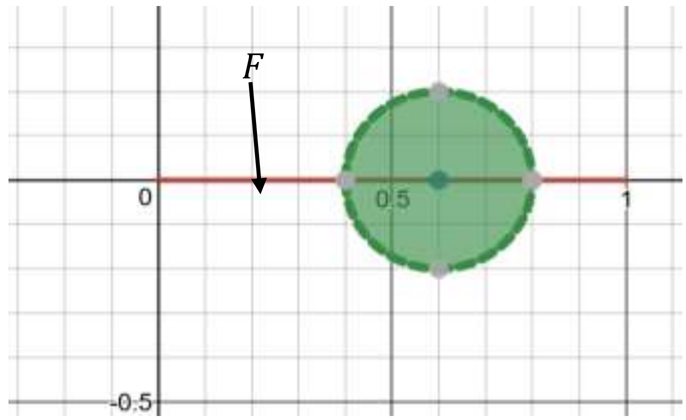
f. E is **open** if every point of E is an interior point

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = (0,1)$.

E is an open set in $X = \mathbb{R}$.

Note: Let $X = \mathbb{R}^2$, and d the standard metric. Let $F = \{(x,y)|\ 0 < x < 1, y = 0\}$

Although F is essentially the same set as E in our example, F is NOT an open subset of $X = \mathbb{R}^2$ because a neighborhood in $\mathbb{R}^2$ is a disk.

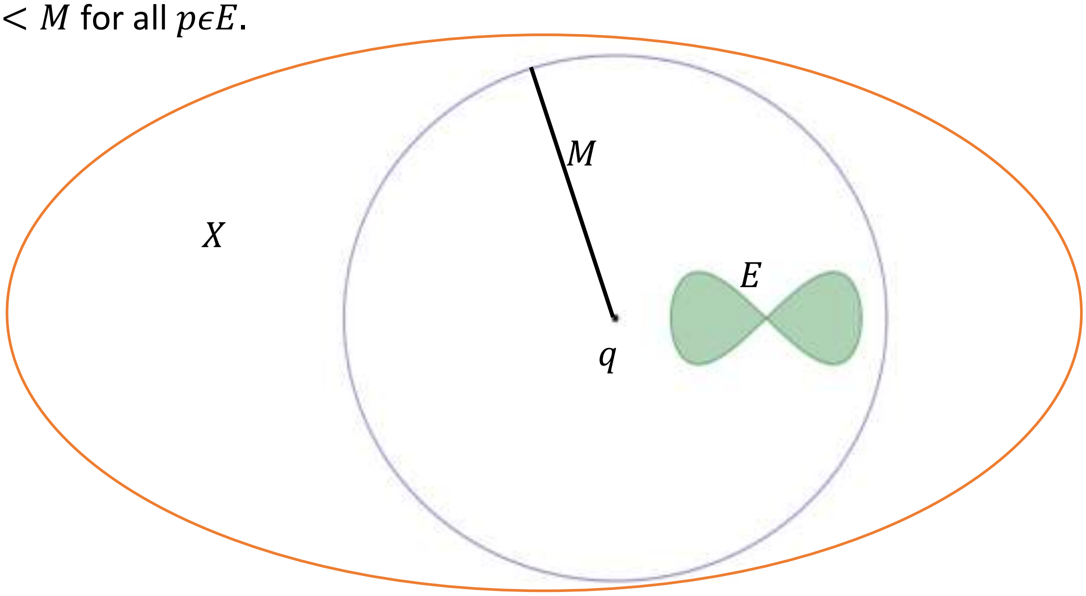


g. The **complement of E** (denoted E^c), is the set of all point $p \epsilon X$ such that $p \notin E$.

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1)$.

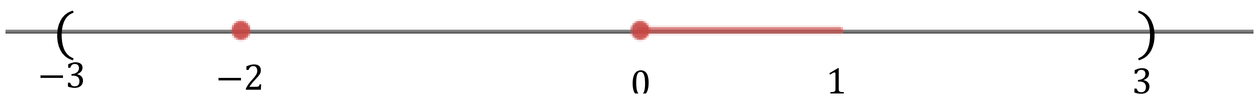
$E^c = (-\infty, 0) \cup [1, \infty)$.

h. E is **bounded** if there is a real number M and a point $q \epsilon X$ such that $d(p,q) < M$ for all $p \epsilon E$.



Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1) \cup \{-2\}$.

E is a bounded set. We can take $0 \epsilon X$ and $d(0, p) < 3$, for all $p \epsilon E$.

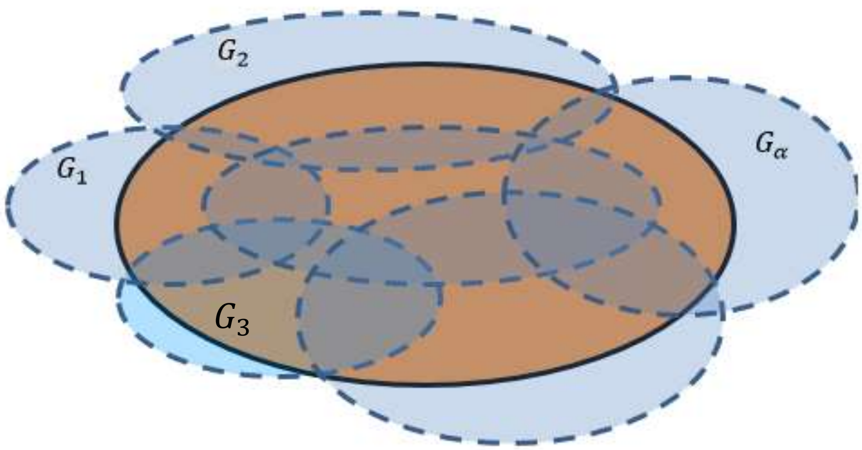


i. E is **dense** in X if every point in X is a limit point of E, or a point of E (or both).

Ex. $E = \bigcup_{i=-\infty}^{i=\infty} (i, i+1)$, E is dense in $X = \mathbb{R}$, and d the standard metric.

Ex. $E = \{rational\ numbers\}$; E is dense in $X = \mathbb{R}$, and d the standard metric.

Def. By an **open cover** of a set $E \subseteq X$, a metric space, we mean a collection $\{G_\alpha\}$ of open sets in X such that $E \subseteq \bigcup_\alpha G_\alpha$



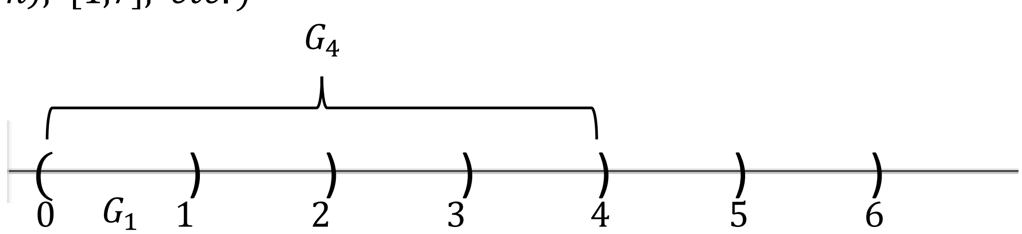
Open Cover of E

Ex. Let $G_i = (0, i) \subseteq \mathbb{R}$. Then $\{G_i\}_{i=1}^{\infty}$ is an open cover of $(0, \infty)$ (it's also an open cover of $(0, n)$, $[1,7]$, $etc.$)

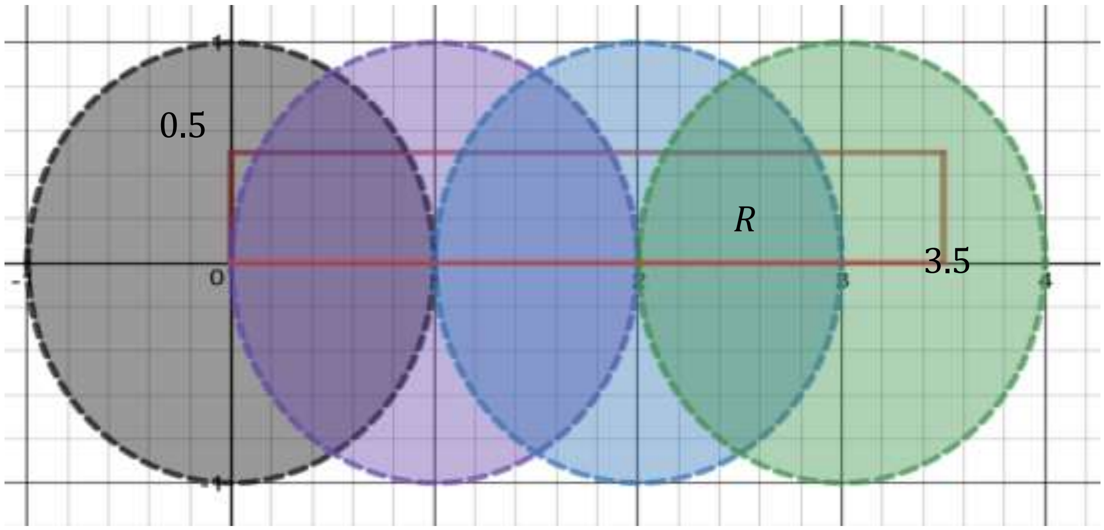
$G_1 = (0,1)$

$G_2 = (0,2)$

$G_3 = (0,3)$

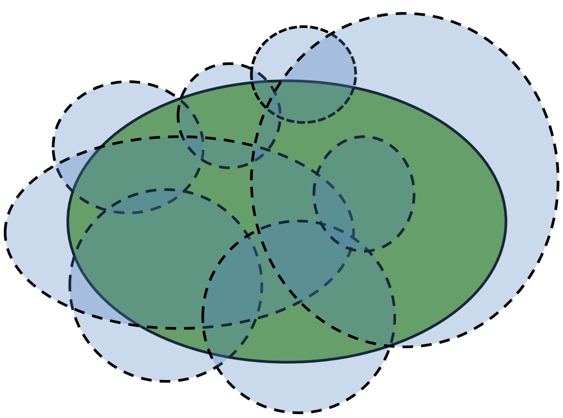


Ex. Let $G_i = \{(x,y)|\ (x-i)^2 + y^2 < 1\},\ \ i = 0,1,2,3.$ $\{G_i\}_{i=0}^{3}$ is an open cover of $R = \{(x,y)|\ 0 \le x \le 3.5,\ \ 0 \le y \le \frac{1}{2}\}$.

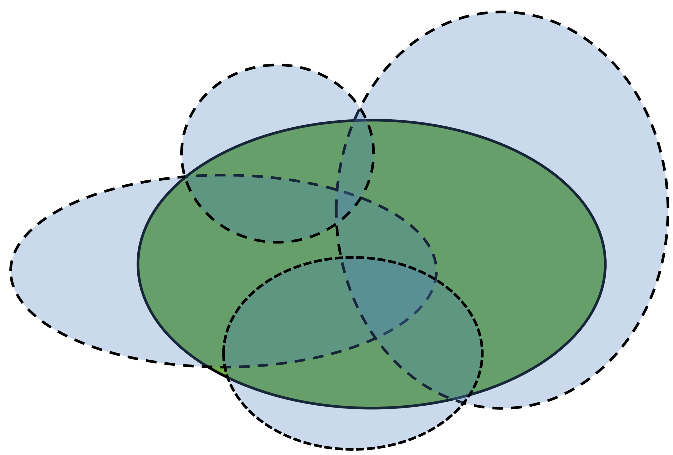


Def. A subset $K \subseteq X, d$ is said to be **compact** if every open cover contains a finite subcover.

This means that if $\{G_\alpha\}$ is any open cover of K, i.e., $K \subseteq \bigcup_\alpha G_\alpha$, then there exist $G_{i_1}, G_{i_2}, \dots, G_{i_n}$ such that $K \subseteq G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n}$.



Open Cover of K



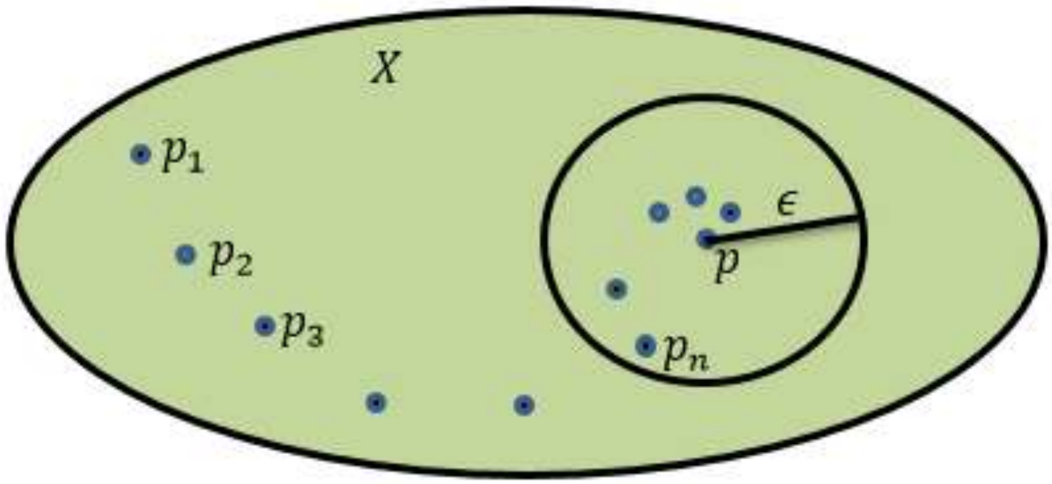
Finite Subcover of K

Heine-Borel Theorem: If E is a set in $\mathbb{R}^n$ then E is compact if and only if E is closed and bounded.

Def. A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is a point $p \epsilon X$ such that for all $\epsilon > 0$, there exists an N, a positive integer, such that if $n \geq N$ then $d(p_n, p) < \epsilon$.

In this case we say that $\lim_{n\to\infty} p_n = p$.

If $\{p_n\}$ does not converge, we say that $\{p_n\}$ **diverges**.



Ex. Prove that the sequence $\{\frac{n}{n+1}\}$ converges to 1, i.e. $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

We must show that given any $\epsilon > 0$ we can find N such that if $n \geq N$ then

$$|p_n - p| = \left|\frac{n}{n+1} - 1\right| < \epsilon.$$

We start with the epsilon statement and try to solve the inequality for n.

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1)}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \epsilon$$

This is equivalent to: $n + 1 > \frac{1}{\epsilon}$

$$n > \frac{1}{\epsilon} - 1.$$

Now we might be tempted to let $N > \frac{1}{\epsilon} - 1$, and that's almost right. We have one small problem. If $\epsilon = 10$, for example, $\frac{1}{\epsilon} - 1$ is a negative number. So just choosing $N > \frac{1}{\epsilon} - 1$ would also include $N = 0$ (but N is a positive integer).

We can get around this problem by letting $N > \max(0, \frac{1}{\epsilon} - 1)$.

Let's show that this choice of N works.

If $n \geq N$ then: $\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{\frac{1}{\epsilon} - 1 + 1} = \epsilon$

So $\lim_{n \to \infty} \frac{n}{n+1} = 1$.

Notice that which metric we use can matter when it comes to convergence.

If we take the sequence $\{\frac{n}{n+1}\}$ but use the metric,

$$\begin{aligned} d(p,q) &= 1 \quad \text{if } p \neq q \\ &= 0 \quad \text{if } p = q \end{aligned}$$

then $d\left(\frac{n}{n+1}, 1\right) = 1$ for all n. Thus with this metric $\{\frac{n}{n+1}\}$ does NOT converge to 1.

Def. A sequence $\{p_n\}$ in a metric space X, d is said to be a **Cauchy Sequence** if for every $\epsilon > 0$ there exists an N, a positive integer, such that if $m, n \geq N$ then $d(p_m, p_n) < \epsilon$.

Theorem: In a metric space X, d, every convergent sequence is a Cauchy sequence.

Note: The converse of this theorem is not true. If $\{p_n\}$ is a Cauchy sequence it does not mean that $p_n \to p$ in X, d. As an example take X ={rational numbers} with the usual metric. Now take a sequence of rational numbers that approaches $\sqrt{2}$, {1, 1.4, 1.41, 1.414, 1.4142, …}. This is a Cauchy sequence but it doesn't converge in X={rational numbers} (although it does converge in the real numbers).

Def. A metric space in which every Cauchy sequence converges is said to be **Complete**.

Ex. $\mathbb{R}^k$ is a complete metric space.

Ex. Prove $\{\frac{1}{n+1}\}$ is a Cauchy sequence in $\mathbb{R}$ (with the usual metric).

We need to show that given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if

$m, n \geq N$ then $d(p_m, p_n) = |\frac{1}{m+1} - \frac{1}{n+1}| < \epsilon.$

By the triangle inequality we have:

$$\left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{m+1} + \frac{1}{n+1}$$

Since $m, n \geq N$ we know that :

$$\left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{m+1} + \frac{1}{n+1} \leq \frac{1}{N+1} + \frac{1}{N+1} = \frac{2}{N+1}\ .$$

So if we can force $\frac{2}{N+1} < \epsilon$, then $\left|\frac{1}{m+1} - \frac{1}{n+1}\right| < \epsilon$.

Solve $\frac{2}{N+1} < \epsilon$ for N

$\frac{N+1}{2} > \frac{1}{\epsilon}$ since both $\frac{2}{N+1}$ and ϵ are positive

$N + 1 > \frac{2}{\epsilon}$

$N > \frac{2}{\epsilon} - 1$.

We have one small technical issue that prevents us from choosing N to be any integer greater than $\frac{2}{\epsilon} - 1$. If $\epsilon = 5$, for example, then $\frac{2}{\epsilon} - 1 < 0$ and thus 0 is an integer greater than $\frac{2}{\epsilon} - 1$. Thus we need to choose N to be any positive integer greater than $\frac{2}{\epsilon} - 1$. We can do that by letting $N > \max\left(0, \frac{2}{\epsilon} - 1\right)$ where N is an integer.

Now let's show that this N "works".

If $m, n \geq N > \max\left(0, \frac{2}{\epsilon} - 1\right)$ then we have:

$$\left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{m+1} + \frac{1}{n+1} \leq \frac{1}{N+1} + \frac{1}{N+1} = \frac{2}{N+1} < \frac{2}{\frac{2}{\epsilon} - 1 + 1} = \epsilon$$

Thus $\{\frac{1}{n+1}\}$ is a Cauchy sequence in $\mathbb{R}$.

Note: As with convergence of sequences, whether a sequence is a Cauchy sequence can depend on which metric you use. In the example above we showed that the sequence $\{\frac{1}{n+1}\}$ is Cauchy using the standard metric however if we take the metric $d(p, q) = |\frac{1}{p} - \frac{1}{q}|$;

$$d\left(\frac{1}{m+1}, \frac{1}{n+1}\right) = |(m + 1) - (n + 1)| = |m - n| \geq 1; \quad \text{if } m \neq n.$$

Thus $\{\frac{1}{n+1}\}$ is NOT a Cauchy sequence with this metric.

However, notice that the sequence $\{n\} = 1, 2, 3, 4, \ldots$ is a Cauchy sequence with this metric since:

$$d(a_n, a_m) = d(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon$$

which can be made less than ϵ by choosing $N > \frac{2}{\epsilon}$.

Thus $[1, \infty]$ is not a complete metric space with this metric.

Def. Let $\{s_n\}$ be a sequence of real numbers such that:

1. If for every real number M there is a positive integer N such that if $n \geq N$ then $s_n \geq M$, then we say $\lim\limits_{n\to\infty} s_n = +\infty$
2. If for every real M there is a positive integer N such that if $n \geq N$ then $s_n \leq M$, then we say $\lim\limits_{n\to\infty} s_n = -\infty$.

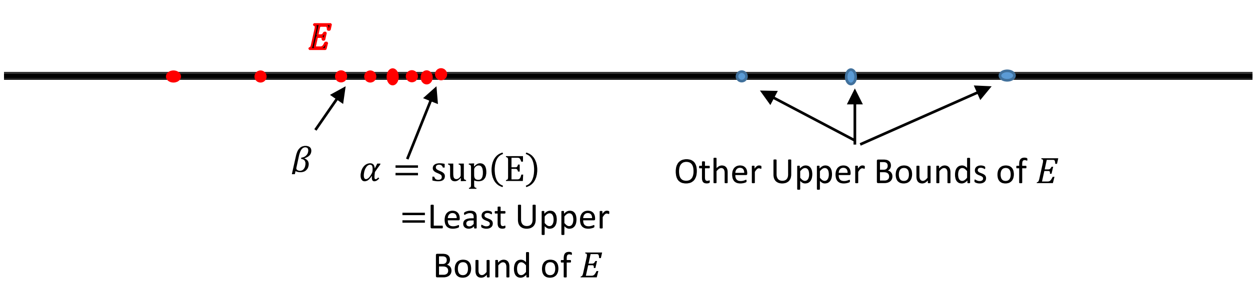
Def. Suppose $E \subseteq \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ and that there exists an $\alpha \epsilon \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ such that:

i. $x \leq \alpha$ for all $x \epsilon E$

ii. if $\beta < \alpha$ then β is not an upper bound for E

then α is called the **Least Upper Bound** for E, or **Supremum** of E, and we write:

$$\alpha = supE.$$

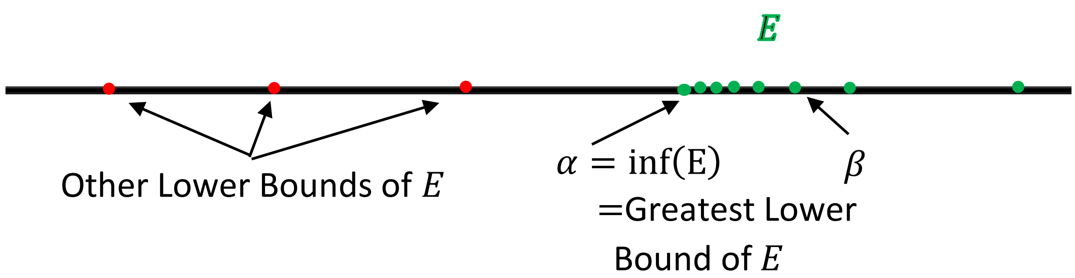


If $\alpha \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ such that:

i. $x \geq \alpha$ for all $x \epsilon E$

ii. if $\beta > \alpha$ then β is not an lower bound for E

then we say α is the **Greatest Lower Bound** for E, or the **Infimum** of E, and we write: $\alpha = inf E$.



Notice that $inf E$ and $sup E$ do not have to lie in E.

Ex. Let $E = (0,1)$.

inf $E = 0$ and sup $E = 1$, neither of which lie in E.

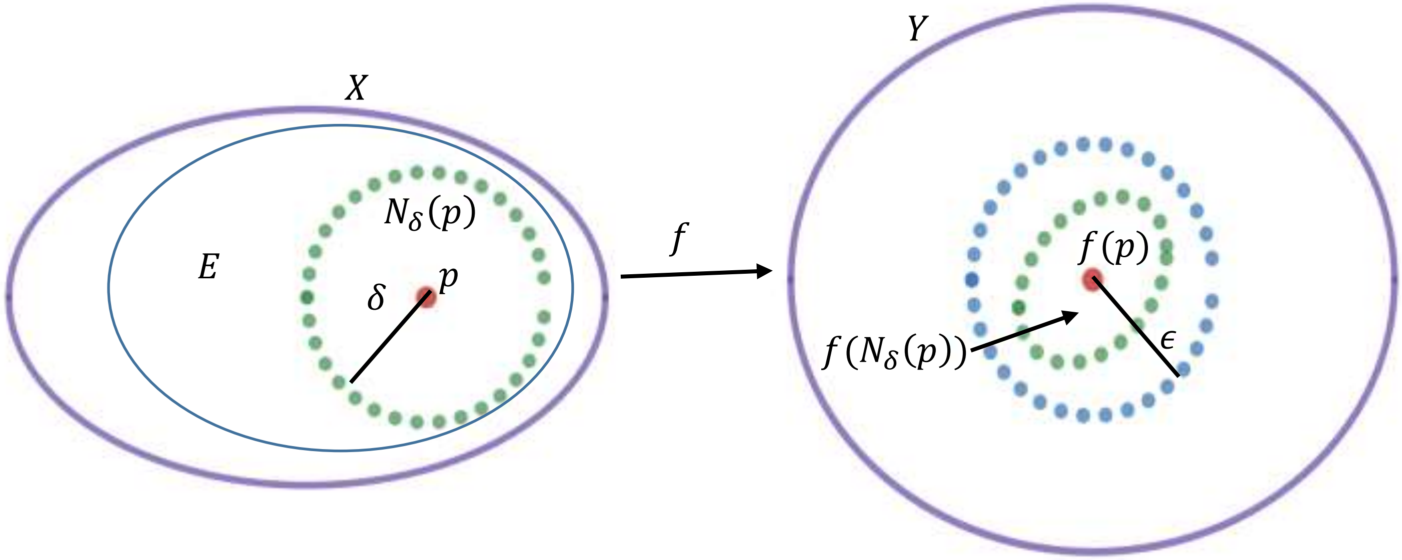
Ex. Let $E = [0, \infty)$

$$inf E = 0, \quad sup E = +\infty$$

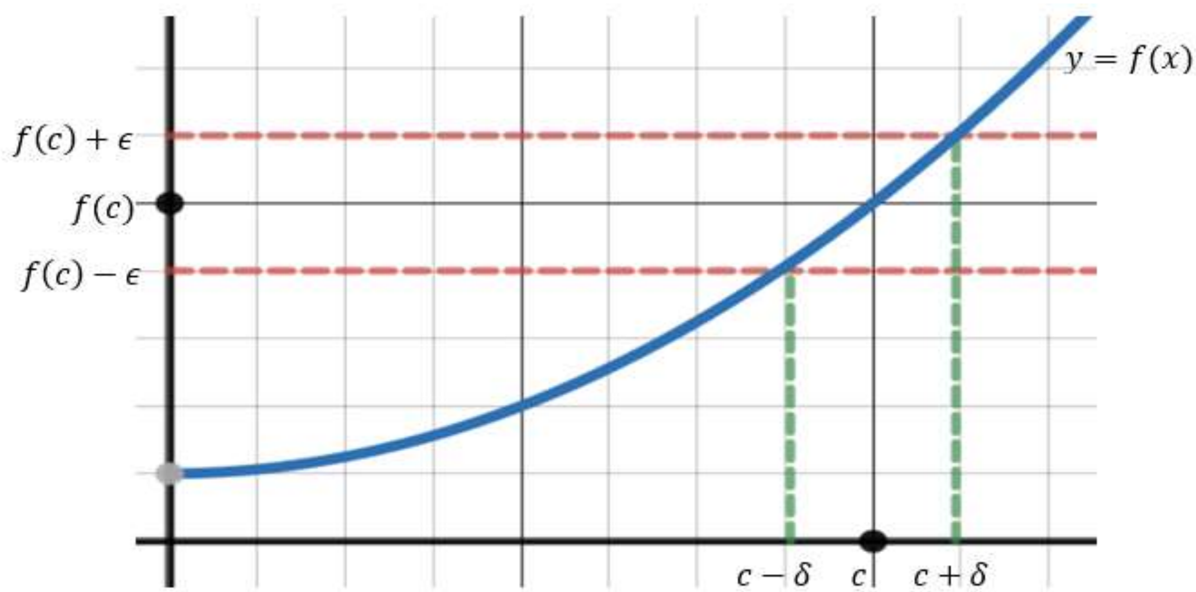
Ex. Let $E = \{x \epsilon \mathbb{R} | \; 2 < x^2 < 3\}$

$$inf E = -\sqrt{3}, \quad sup E = \sqrt{3}$$

Def. Suppose X and Y are metric spaces, $E \subseteq X$, $p \epsilon E$, and $f: E \to Y$. Then f is said to be **Continuous at $\boldsymbol{p}$** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x \epsilon E$, if $d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \epsilon$. Equivalently, we can say that f is **Continuous at $\boldsymbol{p}$** if $\lim_{x \to p} f(x) = f(p)$.



If $X = Y = \mathbb{R}$ then $f(x)$ is **Continuous at $\boldsymbol{x = c}$** means for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.



Def. If f is Continuous at every point of E, then f is said to be **Continuous on $\boldsymbol{E}$**.

Note: For $\lim_{x \to p} f(x)$ to exist, $f(p)$ does not need to be defined (although it can be). For $f(x)$ to be continuous at $p \epsilon E$, $f(p)$ must be defined and equal to $\lim_{x \to p} f(x)$.

Theorem: Suppose f is a continuous function on a compact metric space X into $\mathbb{R}$, and $M = sup_{p \epsilon X} f(p)$ and $m = inf_{p \epsilon X} f(p)$, then there exist points $r, s \epsilon X$ such that $f(r) = M$ and $f(s) = m$.

Def. Let $f: X \to Y$; X, Y are metric spaces. We say f is **uniformly continuous** on X if for every $\epsilon > 0$ there exists a $\delta > 0$ for all $p, q \in X$ such that if $d_X(p,q) < \delta$ then $d_Y\big(f(p), f(q)\big) < \epsilon$.

Notice the difference between continuity and uniform continuity:

1. For uniform continuity, δ does not depend on the point in X you are at. For continuity, the δ can depend on which point in X you are at (with both continuity and uniform continuity, δ does depend on ϵ).

2. Uniform continuity is a property of a set of points, not a single point. Continuity is a property at a point and a set of points.

3. If a function is uniformly continuous on a set X, then it is also continuous on X. However, if a function is continuous on a set X it may, or may not be, uniformly continuous on X.

Ex. Let $f(x) = \frac{1}{x}$; $0 < x < 1$. $f(x)$ is continuous on $(0,1)$ but not uniformly continuous.

To show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

Let's fix an $\epsilon > 0$.

To be uniformly continuous we need to find a $\delta > 0$, that depends only on ϵ, such that if $|x - a| < \delta$ then $\left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon$ for all $a \in (0,1)$.

But if $\epsilon > 0$ is fixed, regardless of what δ one chooses, by moving "a" toward 0

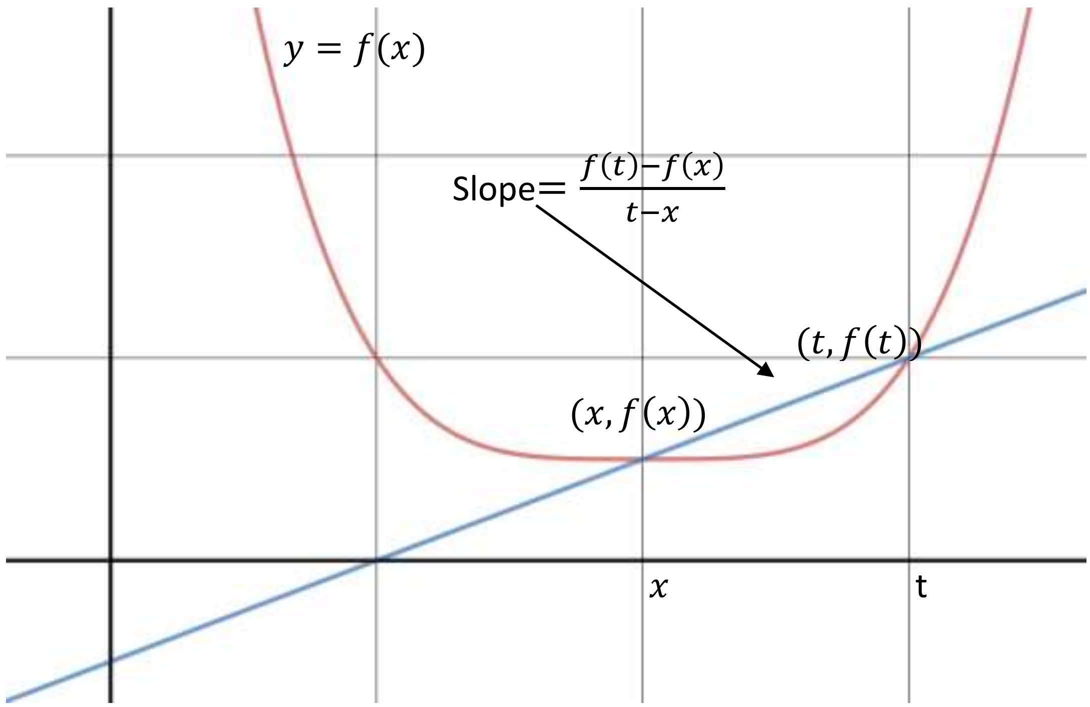
$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{1}{|ax|}|x - a| \to \infty$ for $|x - a| < \delta$.

So δ must depend on "a" and $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

Basic trigonometry will also be important in Analysis 2. You should know the graphs of the the trig functions, the double angle formulas for sin and cos, the sum and difference formulas for sin and cos, the fact that $sin(-x) = -sinx$ and $cos(-x) = cosx$ and the definitions of the inverse trig functions (e.g. $\sin^{-1} a = b$ means that $\sin(b) = a$).

Def. Let f be a real valued function on $[a, b] \subseteq \mathbb{R}$. We define the **derivative of f** at x as: $$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ for } a < t < b, \quad t \neq x$$

if the limit exists.



Notice that we could also say, let $h = t - x$, so that $x + h = t$ and define $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Theorem: Let f be defined on $[a, b]$. If f is differentiable at $x\epsilon[a, b]$ (i.e. $f'(x)$ exists at $x\epsilon[a, b]$) then f is continuous at x.

Proof: To be continuous at x we must show that $\lim_{t\to x} f(t) = f(x)$ or equivalently: $\lim_{t\to x}\big(f(t) - f(x)\big) = 0$.

Notice that $f(t) - f(x) = \left[\frac{(f(t)-f(x))}{t-x}\right](t - x)$; so we have:

$$\lim_{t\to x}\big(f(t) - f(x)\big) = \lim_{t\to x}\left\{\left[\frac{(f(t)-f(x))}{t-x}\right](t - x)\right\}$$

$$= \lim_{t\to x}\left[\frac{(f(t)-f(x))}{t-x}\right]\lim_{t\to x}(t - x)$$

$$= (f'(x))(0) = 0.$$

So differentiability implies continuity, but the converse is not true.

Continuity does not imply differentiability.

Ex. $f(x) = |x|$ is continuous at $x = 0$. Show f is not differentiable at $x = 0$.

$$\lim_{t\to 0^+}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0^+}\frac{t}{t} = 1 \qquad \text{since } f(t) = |t| = t \text{ for } t > 0$$

$$\lim_{t\to 0^-}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0^-}\frac{-t}{t} = -1 \qquad \text{since } f(t) = |t| = -t \text{ for } t < 0.$$

Thus $\lim_{t\to 0}\frac{f(t)-f(0)}{t-0}$ does not exist, so $f'(0)$ does not exist.

It's easy enough to prove that $f(x) = |x|$ is continuous at $x = 0$ so we will skip it here.

In fact it's possible to have a function on $\mathbb{R}$ which is continuous everywhere and differentiable nowhere.

Theorem: $f, g: [a, b] \to \mathbb{R}$ are differentiable at $x\epsilon[a, b]$, then $f \pm g,\ fg, \frac{f}{g}$ (where $g(x) \neq 0$) are differentiable at $x\epsilon[a, b]$ and:

a. $(f \pm g)'(x) = f'(x) \pm g'(x)$

b. $(fg)' = f(x)g'(x) + g(x)f'(x)$

c. $\left(\frac{f}{g}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Theorem (Chain Rule) Suppose f is continuous on $[a, b]$, and $f'(x)$ exists at some point $x\epsilon[a, b]$. Suppose g is defined on an interval which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g\big(f(t)\big)$, $a \leq t \leq b$ then h is differentiable at $t = x$ and $h'(x) = g'(f(x)) \cdot f'(x)$.

Ex. $h(t) = (t^3 + 2t)^9, \implies g(t) = t^9, \quad f(t) = t^3 + 2t$

$h'(x) = g'(f(x)) \cdot f'(x);$

$g'(t) = 9t^8,$ so $g'\big(f(x)\big) = 9(x^3 + 2x)^8; \qquad f'(x) = 3x^2 + 2$

$h'(x) = 9(x^3 + 2x)^8(3x^2 + 2).$

Ex. Let $f(x) = xsin(\frac{1}{x}) \qquad x \neq 0$

$\qquad\qquad = 0 \qquad\qquad\qquad x = 0$

As we saw earlier, $f(x)$ is continuous everywhere (including $x = 0$). Where is $f(x)$ differentiable?

If $x \neq 0$ then we can use the product rule and the chain rule, and we have:

$$f'(x) = x\left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) + \sin(\frac{1}{x})$$

$$= -\frac{1}{x}\left(\cos\left(\frac{1}{x}\right)\right) + \sin\left(\frac{1}{x}\right).$$

At $x = 0$ we have to apply the definition of $f'(0)$ because the chain rule doesn't apply at $x = 0$ since $\frac{1}{x}$ is not differentiable at $x = 0$.

$f'(0) = \lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0} \frac{tsin(\frac{1}{t})}{t} = \lim_{t\to 0} \sin(\frac{1}{t})$; which does not exist.

So $f(x)$ is continuous everywhere and differentiable everywhere except $x = 0$.

Ex. Let $f(x) = x^2sin(\frac{1}{x}) \qquad x \neq 0$

$\qquad\qquad = 0 \qquad\qquad\qquad x = 0.$

Where is $f(x)$ continuous? Where is $f(x)$ differentiable (i.e. $f'(x)$ exists)? Where is $f'(x)$ continuous?

We can show that $f(x)$ is continuous everywhere by showing the $f'(x)$ exists for all $x \in \mathbb{R}$, which is done below.

Where is $f(x)$ differentiable?

If $x \neq 0$ then we can use the product rule and the chain rule, and we have:

$$f'(x) = x^2\left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) + 2x\sin(\frac{1}{x})$$

$$= -\cos\left(\frac{1}{x}\right) + 2x\sin(\frac{1}{x})\ .$$

Notice that $\lim_{x\to 0} f'(x)$ does not exist. Thus, at the very least, $f'(x)$ is not continuous at $x = 0$.

Does $f'(0)$ exist?

At $x = 0$ we have to apply the definition of $f'(0)$ (again, we can't use the chain rule at $x = 0$, since $\frac{1}{x}$ is not differentiable at $x = 0$).

$$f'(0) = \lim_{t\to 0}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0}\frac{t^2 sin(\frac{1}{t})}{t} = \lim_{t\to 0}(\mathrm{t}\, sin(\frac{1}{t})) = 0.$$

We need to justify the last step, $\lim_{t\to 0} tsin\frac{1}{t} = 0.$

Since $|sinx| \leq 1$ for any real number x, we have:

$$0 \leq |tsin\left(\frac{1}{t}\right)| \leq |t|.$$

Since $\lim_{t\to 0} 0 = \lim_{t\to 0}|t| = 0$, by the squeeze theorem $\lim_{t\to 0}|\ tsin\frac{1}{t}\,| = 0.$

Now since $\lim_{t\to a}|f(t)| = 0$ if and only if $\lim_{t\to a} f(t) = 0$, we conclude that

$$\lim_{t\to 0} tsin\frac{1}{t} = 0\ .$$

So $f'(0) = 0$, and $f(x)$ is differentiable everywhere.

We saw that $f'(x)$ is not continuous at $x = 0$. But is $f'(x)$ continuous for $x \neq 0$?

Notice that for $x \neq 0$:

$$f''(x) = -\frac{1}{x^2}\sin\left(\frac{1}{x}\right) + 2\sin\left(\frac{1}{x}\right) - \frac{2}{x}\cos(\frac{1}{x})$$

which is finite for any $x \neq 0$, thus $f'(x)$ is continuous for $x \neq 0$.