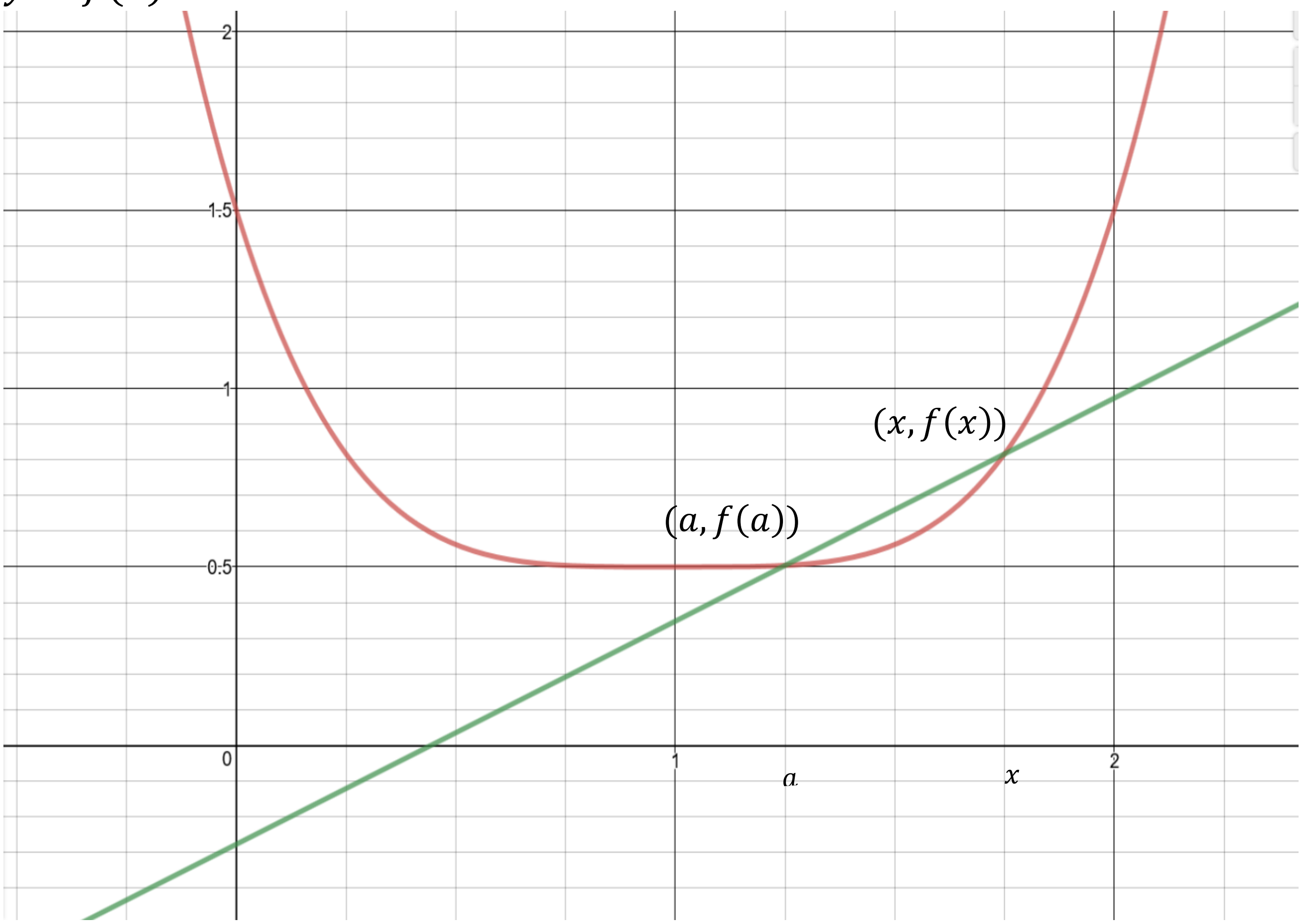
The Derivative of a Function

The derivative of a function, $f(x)$, is another function, $f'(x)$, that equals the slope of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$.

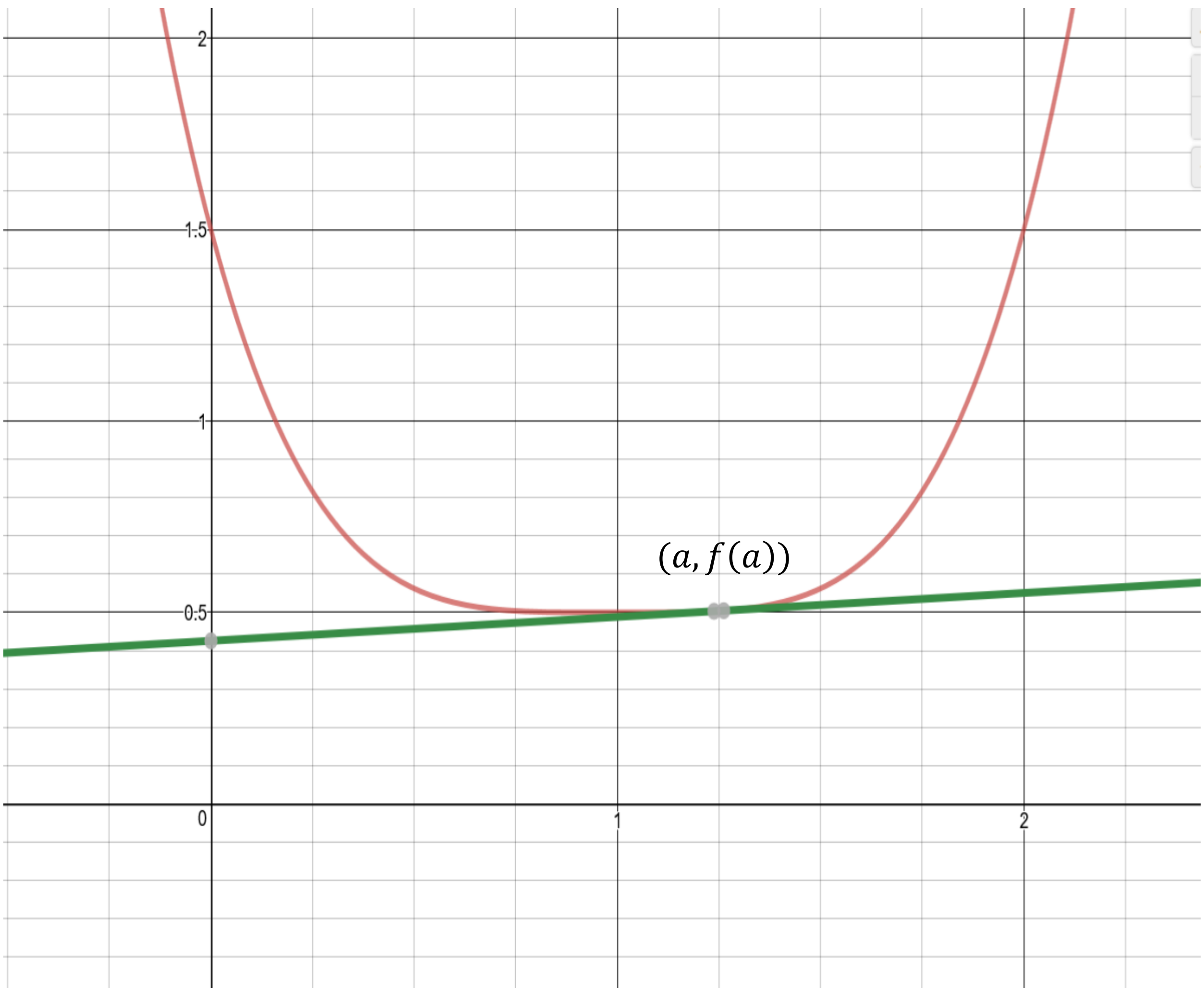
We calculate the slope of the tangent line to $y = f(x)$ at the point $x = a$ by taking the slopes of secant lines between a and x and let x tend toward a. The limit of these slopes (if it exists) is what we call the slope of the tangent line to $y = f(x)$ at $x = a$



Def. $\boldsymbol{f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$ if the limit exists**.**

Notice that $m_{sec} = \frac{f(x)-f(a)}{x-a}$ is the **average rate of change** of the function $y = f(x)$ on the interval $[a, x]$.

$m_{tan} = f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ is the **instantaneous rate of change** of the function $y = f(x)$ at the point $x = a$.



Since $f'(a)$ is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$, we can write down the equation of this tangent line since we know the slope, $f'(a)$, and a point, $(a, f(a))$ on the line.

Equation of tangent line to $y = f(x)$ at $(a, f(a))$:

$$\boldsymbol{y - f(a) = f'(a)(x - a)}.$$

Ex. Find an equation of the tangent line to $y = 4x^2 - 12x$ at the point $(2, -8)$. In this example, $a = 2$.

Slope of the tangent line= $m_{tan} = f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

$$= \lim_{x \to 2} \frac{4x^2 - 12x - (-8)}{x - 2}$$

$$= \lim_{x \to 2} \frac{4x^2 - 12x + 8}{x - 2}$$

$$= \lim_{x \to 2} \frac{4(x^2 - 3x + 2)}{x - 2}$$

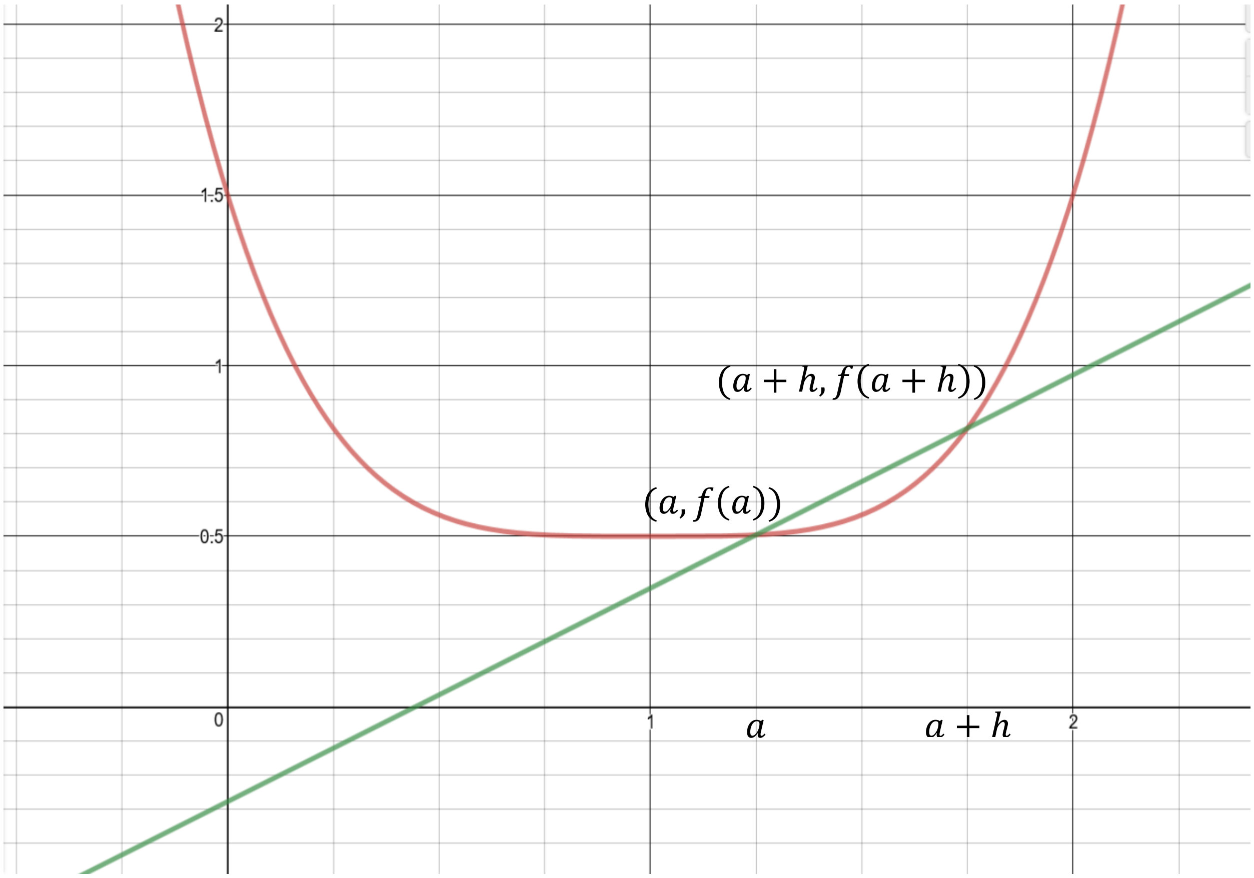
$$= \lim_{x \to 2} \frac{4(x - 2)(x - 1)}{x - 2}$$

$$= \lim_{x \to 2} 4(x - 1) = 4.$$

So the slope of the tangent line at $(2, -8)$ is 4 and an equation of the tangent line is

$$y + 8 = 4(x - 2).$$

An alternative form of the slope of the tangent line can be gotten as follows:



$$\boldsymbol{m_{sec}} = \frac{\boldsymbol{f(a+h) - f(a)}}{\boldsymbol{h}} \quad \text{and} \quad \boldsymbol{m_{tan}} = \boldsymbol{f'(a)} = \lim_{\boldsymbol{h \to 0}} \frac{\boldsymbol{f(a+h) - f(a)}}{\boldsymbol{h}}.$$

Ex. Find the slope of the tangent line to $f(x) = x^3 - 2x + 4$ at the point $(1,3)$. Find and equation for the tangent line at $(1,3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad , \qquad \text{(here } a = 1\text{)}$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^3-2(1+h)+4-3}{h}$$

$$= \lim_{h\to 0} \frac{(1+3h+3h^2+h^3)-2-2h+1}{h}$$

$$= \lim_{h\to 0} \frac{h^3+3h^2+h}{h}$$

$$= \lim_{h\to 0} \frac{h(h^2+3h+1)}{h}$$

$$= \lim_{h\to 0} (h^2 + 3h + 1) = 1.$$

Equation of tangent line at $(1,3)$: $\quad y - f(1) = f'(1)(x-1)$

$$y - 3 = 1(x-1)$$

$$y - 3 = x - 1$$

$$y = x + 2.$$

Notice that we could also have done this problem by saying:

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(1) = \lim_{x \to 1} \frac{x^3-2x+4-3}{x-1}$$

$$= \lim_{x \to 1} \frac{x^3-2x+1}{x-1}$$

$$= \lim_{x \to 1} \frac{(x^2+x-1)(x-1)}{x-1}$$

$$= \lim_{x \to 1}(x^2 - x + 1) = 1^2 - 1 + 1 = 1.$$

Ex. If $f'(a) = \lim_{h \to 0} \frac{4(2+h)^3-32}{h}$, what is $f(x)$ and what is a?

$$f(x) = 4x^3, \qquad a = 2.$$

<u>The Derivative as a Function</u>

So far we have calculated the derivative of a function at a fixed point $x = a$. However, for each point x in the domain of $f(x)$ we can ask if the graph of $f(x)$ has a unique tangent line at that point and hence $f'(x)$ exists at that point.

Def. The derivative function $f'(x)$ is defined as

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}.$$

if this limit exists.

If $f(x)$ has a derivative at x we say $f(x)$ is **differentiable at $\boldsymbol{x}$**. If $f'(x)$ exists for every x in an interval I we say $f(x)$ is **differentiable on $\boldsymbol{I}$**.

Ex. Let $f(x) = 4x^2 - 12x$, calculate $f'(x)$ when x is any real number.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4(x+h)^2-12(x+h)-(4x^2-12x)}{h}$$

$$= \lim_{h\to 0} \frac{4\left(x^2+2hx+h^2\right)-12x-12h-4x^2+12x}{h}$$

$$= \lim_{h\to 0} \frac{4x^2+8hx+4h^2-12h-4x^2}{h}$$

$$= \lim_{h\to 0} \frac{8hx+4h^2-12h}{h} = \lim_{h\to 0} \frac{h(8x+4h-12)}{h}$$

$$= \lim_{h\to 0}(8x + 4h - 12) = 8x - 12.$$

So $f'(x) = 8x - 12$.

Notice that if we plug in $x = 2$ we get: $f'(2) = 8(2) - 12 = 4$, which is the same answer that we got in the first example in this section.

Notation

There are several common notations for the derivative of a function. So far we have used $f'(x)$. Other common notations are;

$$\frac{dy}{dx}, \quad \frac{d}{dx}\big(f(x)\big), \quad \frac{df}{dx}, \quad D_x\big(f(x)\big), \quad y'(x).$$

If we are evaluating the derivative at $x = a$ we can write,

$$f'(a), \quad y'(a), \quad \frac{dy}{dx}\Big|_{x=a}, \quad \frac{df}{dx}\Big|_{x=a}.$$

Ex. Let $y = \sqrt{x}$

a. Compute $\frac{dy}{dx}$
b. Find an equation of the tangent line to the graph of $y = \sqrt{x}$ at $(9,3)$.

a. $$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$
$$= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$
$$= \lim_{h\to 0}\left(\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\right)$$
$$= \lim_{h\to 0}\left(\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right)$$
$$= \lim_{h\to 0}\left(\frac{1}{(\sqrt{x+h}+\sqrt{x})}\right) = \frac{1}{2\sqrt{x}}.$$

b. The slope of the tangent line at $(9,3)$ is $\frac{dy}{dx}$ when $x = 9$, i.e., $\frac{1}{2\sqrt{9}} = \frac{1}{6}$.

Hence an equation of the tangent line is: $y - 3 = \frac{1}{6}(x-9)$.

Ex. Let $g(t) = \frac{1}{t}$, find $g'(t)$ and an equation of the tangent line at $\left(5,\frac{1}{5}\right)$.

$$g'(t) = \lim_{h\to 0}\frac{g(t+h)-g(t)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{1}{t+h}-\frac{1}{t}}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{t+h}-\frac{1}{t}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} [\frac{t-(t+h)}{(t+h)(t)}]$$
$$= \lim_{h \to 0} \frac{1}{h} [\frac{t-t-h}{(t+h)(t)}]$$
$$= \lim_{h \to 0} \frac{1}{h} [\frac{-h}{(t+h)(t)}]$$
$$= \lim_{h \to 0} [\frac{-1}{(t+h)(t)}] = -\frac{1}{t^2}.$$

So $g'(t) = -\frac{1}{t^2}$.

Slope of tangent line at $\left(5, \frac{1}{5}\right)$ is $g'(5) = -\frac{1}{5^2} = -\frac{1}{25}$.

Equation of tangent line at $\left(5, \frac{1}{5}\right)$: $y - \frac{1}{5} = -\frac{1}{25}(x-5)$.

Ex. Let $f(x) = \frac{12}{x^2}$. Find $f'(x)$ and an equation for the tangent line at $(2,3)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{12}{(x+h)^2} - \frac{12}{x^2}}{h}$$
$$= \lim_{h \to 0} \frac{1}{h} (\frac{12}{(x+h)^2} - \frac{12}{x^2})$$
$$= \lim_{h \to 0} \frac{12}{h} (\frac{x^2-(x+h)^2}{(x+h)^2 x^2})$$
$$= \lim_{h \to 0} \frac{12}{h} (\frac{x^2-(x^2+2hx+h^2)}{(x+h)^2 x^2})$$
$$= \lim_{h \to 0} \frac{12}{h} (\frac{-2hx-h^2}{(x+h)^2 x^2})$$
$$= \lim_{h \to 0} \frac{12}{h} (\frac{h(-2x-h)}{(x+h)^2 x^2})$$
$$= \lim_{h \to 0} 12 \left(\frac{(-2x-h)}{(x+h)^2 x^2}\right) = -\frac{24x}{x^4} = -\frac{24}{x^3}.$$

Slope of tangent line at $(2,3)$ is $f'(2) = -\frac{24}{2^3} = -3$.

Equation of tangent line at $(2,3)$: $y - 3 = -3(x - 2)$.