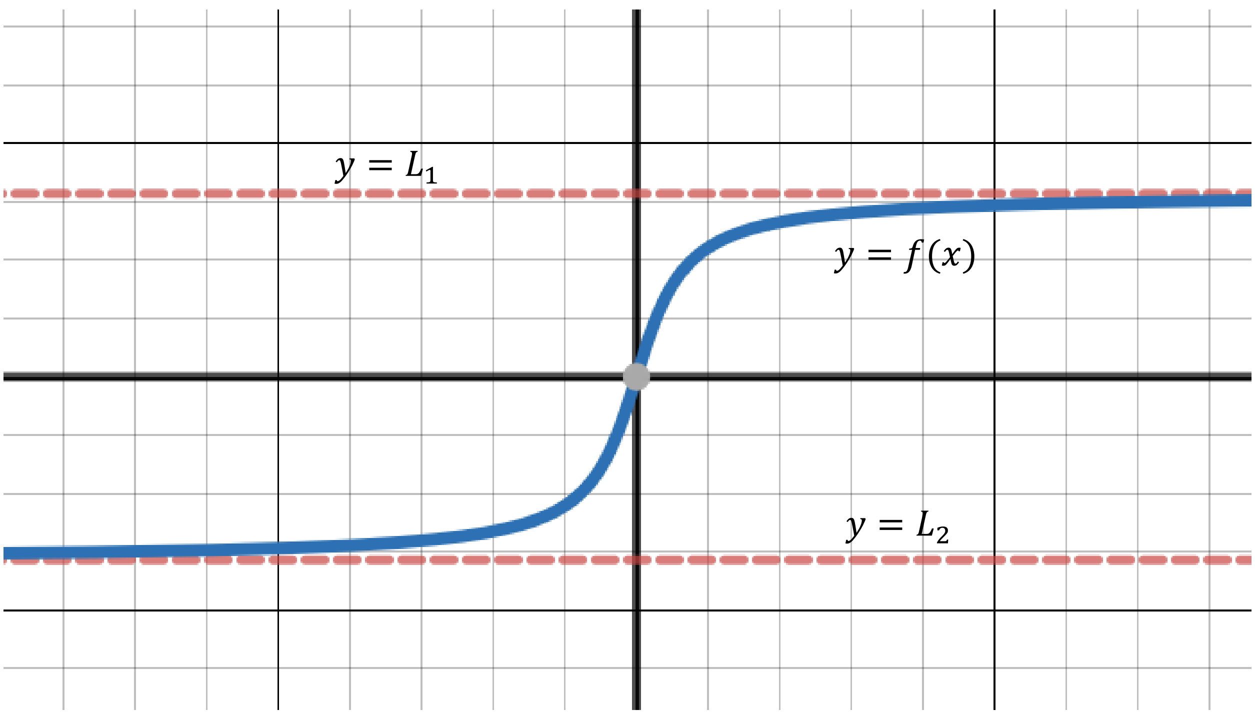
Limits at Infinity

Limits at infinity occur when x (or the independent variable) becomes very large in magnitude. These limits determine the **end behavior** of a function.

Informal definition: $\lim\limits_{x\to\infty} f(x) = L$ means as x goes toward ∞ the value of $f(x)$ goes toward L.

Similarly, $\lim\limits_{x\to-\infty} f(x) = L$ means as x goes toward $-\infty$ the value of $f(x)$ goes toward L.

Ex. $\lim\limits_{x\to\infty} f(x) = L_1$ and $\lim\limits_{x\to-\infty} f(x) = L_2$.



Def. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$ the line $y = L$ is called a **horizontal asymptote** for the graph of the function $y = f(x)$.

Ex. For any positive integer m, $y = \frac{1}{x^m}$ has a horizontal asymptote at $y = 0$ since as x goes to either ∞ or $-\infty$, $\frac{1}{x^m}$ goes toward 0 (i.e. $\lim_{x\to\infty} \frac{1}{x^m} = 0$ and $\lim_{x\to-\infty} \frac{1}{x^m} = 0$).

If m is any positive real number then $\lim_{x\to\infty} \frac{1}{x^m} = 0$. $\lim_{x\to-\infty} \frac{1}{x^m}$ may or may not exist. For example, $\lim_{x\to-\infty} \frac{1}{x^{\frac{1}{2}}} = \lim_{x\to-\infty} \frac{1}{\sqrt{x}}$ doesn't exist since $\frac{1}{\sqrt{x}}$ is not defined for $x < 0$.

Ex. Evaluate the following limits:

a. $\lim_{x\to-\infty} (4 - \frac{3}{x^2})$
b. $\lim_{x\to\infty} (3 + \frac{cosx}{\sqrt{x}})$

a. By our limits laws:

$$\lim_{x\to-\infty} (4 - \frac{3}{x^2}) = \lim_{x\to-\infty} 4 - \lim_{x\to-\infty} \frac{3}{x^2}$$
$$= \lim_{x\to-\infty} 4 - (3)(\lim_{x\to-\infty} \frac{1}{x^2}) = 4 - 3(0) = 4.$$

b. Notice that $-1 \le cosx \le 1$ for all real numbers x so

$$\frac{-1}{\sqrt{x}} \le \frac{cosx}{\sqrt{x}} \le \frac{1}{\sqrt{x}}; \text{ for } x > 0.$$

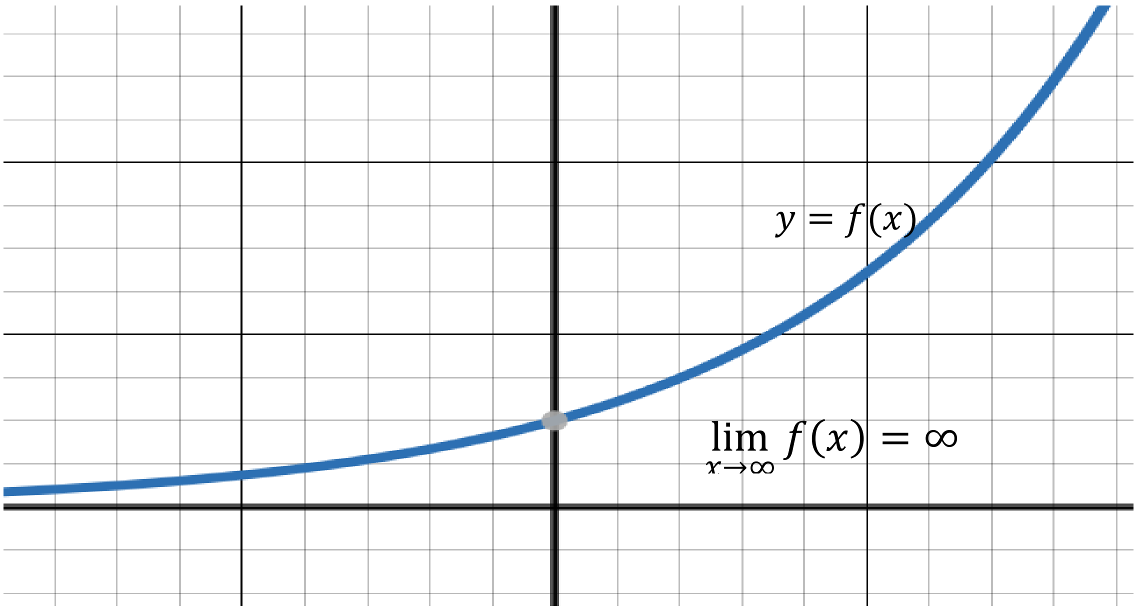
By the squeeze theorem since $\lim_{x\to\infty} \frac{-1}{\sqrt{x}} = 0$ and $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$,

$$\Rightarrow \quad \lim_{x\to\infty} \frac{cosx}{\sqrt{x}} = 0.$$

Thus $\lim_{x\to\infty} (3 + \frac{cosx}{\sqrt{x}}) = \lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{cosx}{\sqrt{x}} = 3 + 0 = 3.$

Infinite Limits at infinity

Informal definition: If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write $\lim_{x\to\infty} f(x) = \infty$. $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined analogously.

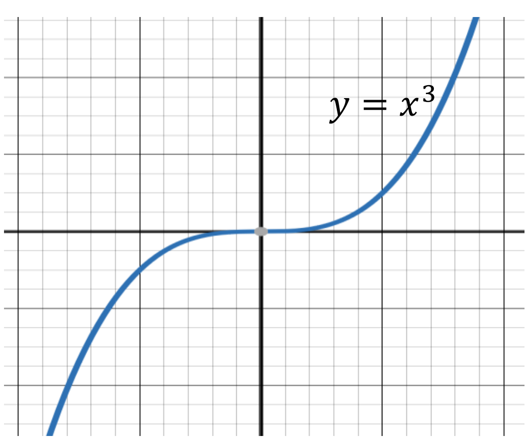


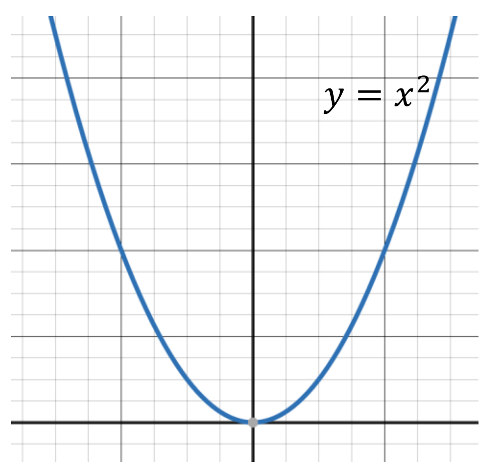
Ex. $\lim_{x\to\infty} x^3 = \infty$, $\lim_{x\to-\infty} x^3 = -\infty$

$\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to-\infty} x^2 = \infty$.

In fact: $\lim_{x\to\infty} x^n = \infty$, $\lim_{x\to-\infty} x^n = -\infty$, if n is a positive odd number

$\lim_{x\to\infty} x^n = \infty$, $\lim_{x\to-\infty} x^n = \infty$, if n is a positive even number.





The end behavior of a polynomial is determined by whether the degree of the highest power is odd or even AND the sign of the coefficient of that term.

$$p(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$
$$= x^n\left(b_n + \frac{b_{n-1}}{x} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}\right)$$

As x goes to $\pm\infty$ all of the terms in the parentheses go to 0 except the first one. So as x goes to $\pm\infty$, $p(x)$ has end behavior of $b_n x^n$.

Ex. Find $\lim_{x\to\infty}(-4x^{15} + 100x^{14} - 3x^7 + 3)$ and $\lim_{x\to-\infty}(-4x^{15} + 100x^{14} - 3x^7 + 3)$.

$$\lim_{x\to\infty}(-4x^{15} + 100x^{14} - 3x^7 + 3) = \lim_{x\to\infty}(-4x^{15}) = -\infty$$

since $\lim_{x\to\infty} x^{15} = \infty$.

$$\lim_{x\to-\infty}(-4x^{15} + 100x^{14} - 3x^7 + 3) = \lim_{x\to-\infty}(-4x^{15}) = \infty$$

since $\lim_{x\to-\infty} x^{15} = -\infty$.

<u>End Behavior of Rational Functions and Algebraic Functions</u>

To determine the end behavior of a rational function $\frac{p(x)}{q(x)}$, divide the numerator and the denominator by the highest power in the denominator.

Ex. Determine the end behavior of:

a. $\frac{3x-1}{2x^3+x}$

b. $\frac{-2x^4+x^2+3}{-2x^3+x-1}$

c. $\frac{10x^4+3x-2}{-2x^4+x^2-1}$

a. $\frac{3x-1}{2x^3+x} = \frac{x^3(\frac{3}{x^2}-\frac{1}{x^3})}{x^3(2+\frac{1}{x^2})} = \frac{(\frac{3}{x^2}-\frac{1}{x^3})}{(2+\frac{1}{x^2})}$; so

$$\lim_{x\to\infty}\frac{3x-1}{2x^3+x} = \lim_{x\to\infty}\frac{(\frac{3}{x^2}-\frac{1}{x^3})}{(2+\frac{1}{x^2})} = \frac{0-0}{2+0} = \frac{0}{2} = 0$$

$$\lim_{x\to-\infty}\frac{3x-1}{2x^3+x} = \lim_{x\to-\infty}\frac{(\frac{3}{x^2}-\frac{1}{x^3})}{(2+\frac{1}{x^2})} = \frac{0-0}{2+0} = \frac{0}{2} = 0.$$

So $y = 0$ is a horizontal asymptote for this function.

b. $$\frac{-2x^4+x^2+3}{-2x^3+x-1} = \frac{x^3(-2x+\frac{1}{x}+\frac{3}{x^3})}{x^3(-2+\frac{1}{x^2}-\frac{1}{x^3})} = \frac{(-2x+\frac{1}{x}+\frac{3}{x^3})}{(-2+\frac{1}{x^2}-\frac{1}{x^3})};\ \text{so}$$

$$\lim_{x\to\infty} \frac{-2x^4+x^2+3}{-2x^3+x-1} = \lim_{x\to\infty} \frac{(-2x+\frac{1}{x}+\frac{3}{x^3})}{(-2+\frac{1}{x^2}-\frac{1}{x^3})} = \lim_{x\to\infty} \frac{-2x}{-2} = \lim_{x\to\infty} x = \infty.$$

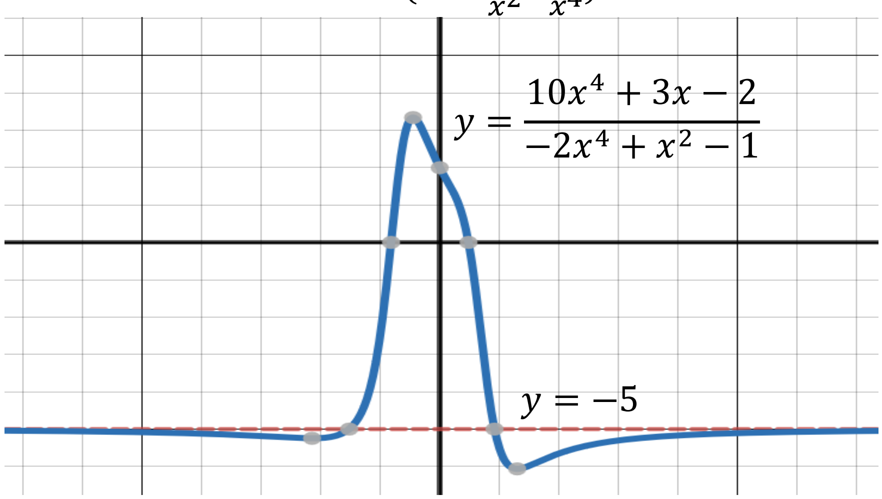
$$\lim_{x\to-\infty} \frac{-2x^4+x^2+3}{-2x^3+x-1} = \lim_{x\to-\infty} \frac{(-2x+\frac{1}{x}+\frac{3}{x^3})}{(-2+\frac{1}{x^2}-\frac{1}{x^3})} = \lim_{x\to-\infty} \frac{-2x}{-2} = \lim_{x\to-\infty} x = -\infty.$$

So no horizontal asymptotes for this function.

c. $$\frac{10x^4+3x-2}{-2x^4+x^2-1} = \frac{x^4(10+\frac{3}{x^3}-\frac{2}{x^4})}{x^4(-2+\frac{1}{x^2}-\frac{1}{x^4})} = \frac{(10+\frac{3}{x^3}-\frac{2}{x^4})}{(-2+\frac{1}{x^2}-\frac{1}{x^4})};\ \text{so}$$

$$\lim_{x\to\infty} \frac{10x^4+3x-2}{-2x^4+x^2-1} = \lim_{x\to\infty} \frac{(10+\frac{3}{x^3}-\frac{2}{x^4})}{(-2+\frac{1}{x^2}-\frac{1}{x^4})} = \frac{10}{-2} = -5.$$

$$\lim_{x\to-\infty} \frac{10x^4+3x-2}{-2x^4+x^2-1} = \lim_{x\to-\infty} \frac{(10+\frac{3}{x^3}-\frac{2}{x^4})}{(-2+\frac{1}{x^2}-\frac{1}{x^4})} = \frac{10}{-2} = -5.$$



So $y = -5$ is a horizontal asymptote for this function.

Summary of End Behavior and Asymptotes of Rational Functions:

If $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ is a polynomial of degree r and $q(x)$ is a polynomial of degree s where:

$$p(x) = c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0; \qquad c_r \neq 0$$
$$q(x) = d_s x^s + d_{s-1}x^{s-1} + \cdots + d_1 x + d_0; \qquad d_s \neq 0.$$

1. If the degree of the numerator is less than the degree of the denominator, $r < s$, then $\lim_{x \to \pm\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote of $f(x)$.

2. If the degree of the numerator equals the degree of the denominator, $r = s$, then $\lim_{x \to \pm\infty} f(x) = \frac{c_r}{d_s}$ and $y = \frac{c_r}{d_s}$ is a horizontal asymptote of $f(x)$.

3. If the degree of the numerator is greater than the degree of the denominator, $r > s$, then $\lim_{x \to \pm\infty} f(x) = \infty \; or - \infty$, and $f(x)$ has no horizontal asymptote.

4. Assuming $f(x)$ is in reduced form, vertical asymptotes occur at the zeros of $q(x)$.

End Behavior for Algebraic Functions

Ex. Evaluate: $\lim_{x\to\pm\infty} \frac{6x+1}{\sqrt{4x^2+3x+5}}$

The highest power in the denominator is $\sqrt{x^2} = x$ when x is positive:

$$\lim_{x\to\infty} \frac{6x+1}{\sqrt{4x^2+3x+5}} = \lim_{x\to\infty} \frac{x(6+\frac{1}{x})}{\sqrt{x^2(4+\frac{3}{x}+\frac{5}{x^2})}}$$

$$= \lim_{x\to\infty} \frac{x(6+\frac{1}{x})}{\sqrt{x^2}\sqrt{4+\frac{3}{x}+\frac{5}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{x(6+\frac{1}{x})}{x\sqrt{4+\frac{3}{x}+\frac{5}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{6+\frac{1}{x}}{\sqrt{4+\frac{3}{x}+\frac{5}{x^2}}}$$

$$= \frac{6}{\sqrt{4}} = 3.$$

When x is negative $\sqrt{x^2} = -x$:

$$\lim_{x\to-\infty} \frac{6x+1}{\sqrt{4x^2+3x+5}} = \lim_{x\to-\infty} \frac{x(6+\frac{1}{x})}{\sqrt{x^2(4+\frac{3}{x}+\frac{5}{x^2})}}$$

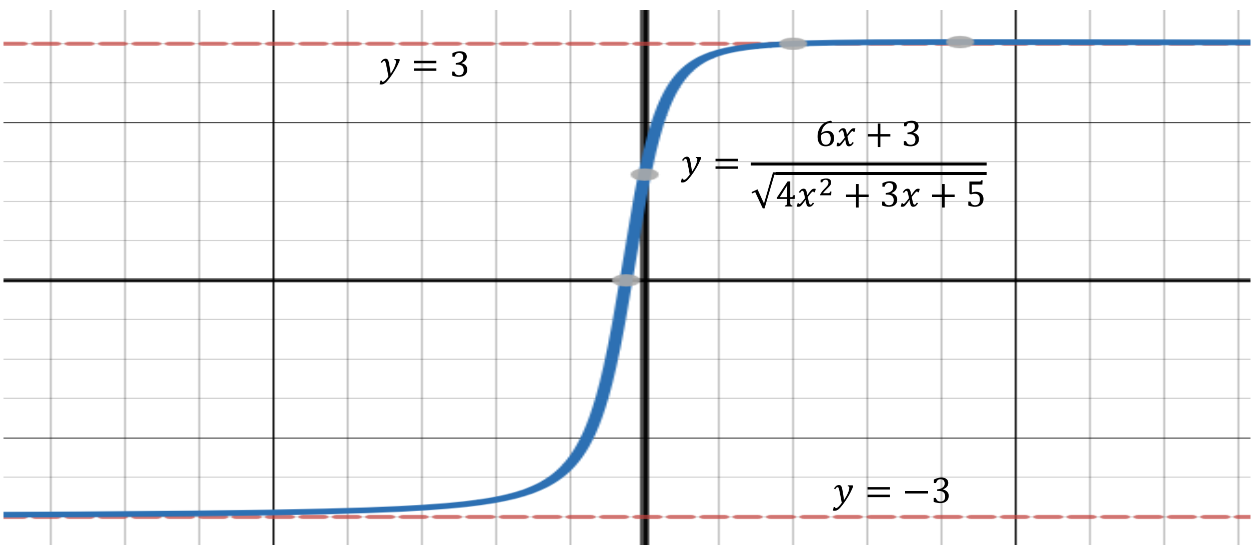
$$= \lim_{x\to-\infty} \frac{x(6+\frac{1}{x})}{\sqrt{x^2}\sqrt{4+\frac{3}{x}+\frac{5}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{x(6+\frac{1}{x})}{-x\sqrt{4+\frac{3}{x}+\frac{5}{x^2}}} \; ; \qquad \sqrt{x^2} = -x \text{ since } x < 0.$$

$$= \lim_{x \to -\infty} -\frac{6+\frac{1}{x}}{\sqrt{4+\frac{3}{x}+\frac{5}{x^2}}}$$

$$= -\frac{6}{\sqrt{4}} = -3.$$

So $f(x) = \frac{6x+3}{\sqrt{4x^2+3x+5}}$ has horizontal asymptotes at $y = \pm 3$.



Ex. Determine the end behavior of $f(x) = \frac{\sqrt{x^6+2}}{3x^3+2x}$.

As with rational functions we want to divide the numerator and the denominator by the "highest power" in the denominator. However, since there is a square root in the numerator we have to be careful. The highest power in the numerator is actually $\sqrt{x^6} = x^3$ if $x > 0$ and $\sqrt{x^6} = -x^3$ if $x < 0$. Either way, the "highest power" in the numerator is 3, the same as the denominator.

For $x > 0$, $\sqrt{x^6} = x^3$,

$$\lim_{x\to\infty} \frac{\sqrt{x^6+2}}{3x^3+2x} = \lim_{x\to\infty} \frac{\sqrt{x^6(1+\frac{2}{x^6})}}{x^3(3+\frac{1}{x^2})} = \lim_{x\to\infty} \frac{\sqrt{x^6}\sqrt{(1+\frac{2}{x^6})}}{x^3(3+\frac{1}{x^2})}$$

$$= \lim_{x\to\infty} \frac{x^3\sqrt{(1+\frac{2}{x^6})}}{x^3(3+\frac{1}{x^2})} = \lim_{x\to\infty} \frac{\sqrt{(1+\frac{2}{x^6})}}{(3+\frac{1}{x^2})} = \frac{1}{3}\,.$$

For $x < 0$, $\sqrt{x^6} = -x^3$,

$$\lim_{x\to-\infty} \frac{\sqrt{x^6+2}}{3x^3+2x} = \lim_{x\to-\infty} \frac{\sqrt{x^6(1+\frac{2}{x^6})}}{x^3(3+\frac{1}{x^2})} = \lim_{x\to-\infty} \frac{\sqrt{x^6}\sqrt{(1+\frac{2}{x^6})}}{x^3(3+\frac{1}{x^2})}$$

$$= \lim_{x\to-\infty} \frac{-x^3\sqrt{(1+\frac{2}{x^6})}}{x^3(3+\frac{1}{x^2})} = \lim_{x\to-\infty} \frac{-\sqrt{(1+\frac{2}{x^6})}}{(3+\frac{1}{x^2})} = \frac{-1}{3}\,.$$

So $y = \frac{1}{3}$ is a horizontal asymptote and $y = -\frac{1}{3}$ is a horizontal asymptote.

Note: If the highest power under the square root was a multiple of 4, like x^8, we wouldn't have had to worry about the sign of the radical because for any value of x, $\sqrt{x^8} = x^4$.

End Behavior of $sinx$ and $cosx$

$sinx$ and $cosx$ oscillate so $\lim_{x\to\pm\infty} sinx$ and $\lim_{x\to\pm\infty} cosx$ do not exist.