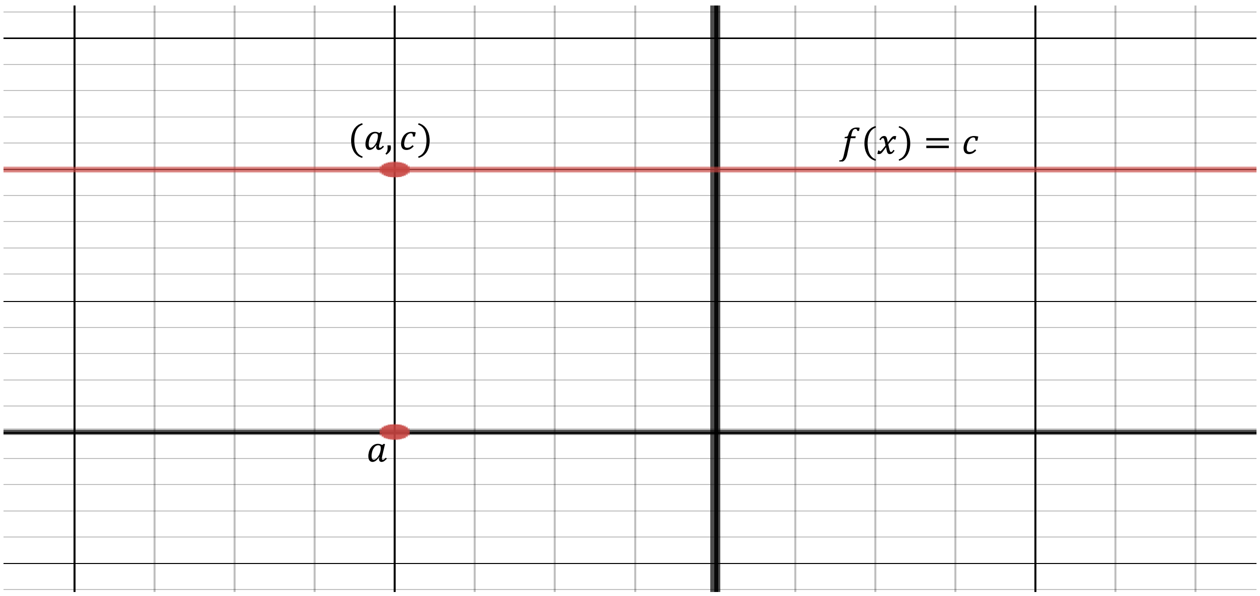
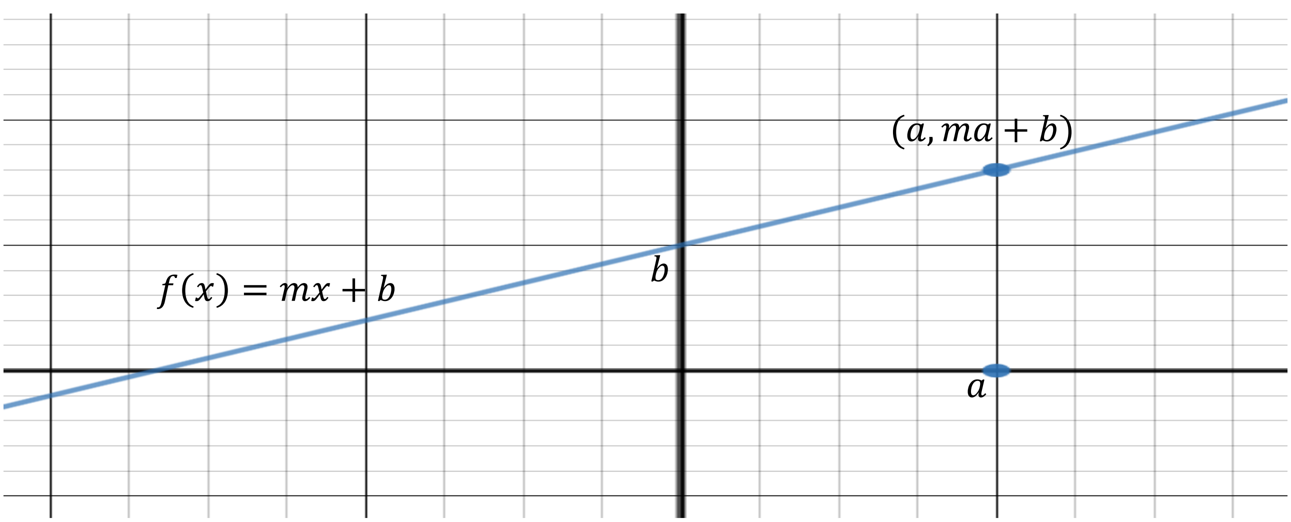
Calculating Limits

$\lim_{x \to a} f(x) = c$ where $f(x) = c$ is a constant function.



$\lim_{x \to a} f(x) = ma + b$ where $f(x) = mx + b$.



Ex. Find $\lim_{x \to 2} 9$, $\lim_{x \to -3} (-2x + 4)$.

$\lim_{x \to 2} 9 = 9$ since $f(x) = 9$ is a constant function.

$\lim_{x \to -3} (-2x + 4) = -2(-3) + 4 = 10.$

Limit Laws: Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then the following relationships hold, where c is a real number, and m, n are positive integers.

1. Sum: $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. Difference: $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. Constant Multiple: $\lim_{x \to a} (c\, f(x)) = c \lim_{x \to a} f(x)$
4. Product: $\lim_{x \to a} (f(x) g(x)) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$
5. Quotient: $\lim_{x \to a} (\frac{f(x)}{g(x)}) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, as long as $\lim_{x \to a} g(x) \neq 0$
6. Power: $\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n$
7. Fractional Power: $\lim_{x \to a} (f(x))^{\frac{n}{m}} = (\lim_{x \to a} f(x))^{\frac{n}{m}}$; provided

   $f(x) > 0,$ for x near a, if m is even and n/m is reduced to lowest form.

Ex. Evaluate $\lim_{x\to 3}(2x^2 - x + 4)$.

$$\lim_{x\to 3}(2x^2 - x + 4) = \lim_{x\to 3} 2x^2 - \lim_{x\to 3} x + \lim_{x\to 3} 4 \qquad \text{(by laws 1 and 2)}$$

$$= 2\lim_{x\to 3} x^2 - \lim_{x\to 3} x + \lim_{x\to 3} 4 \qquad \text{(by law 3)}$$

$$= 2(\lim_{x\to 3} x)^2 - \lim_{x\to 3} x + \lim_{x\to 3} 4 \qquad \text{(by law 6)}$$

$$= 2(3)^2 - 3 + 4 = 18 - 3 + 4 = 19.$$

Ex. Evaluate $\lim_{x\to 2}(x^3 - 4x^2 + 1)$.

$$\lim_{x\to 2}(x^3 - 4x^2 + 1) = \lim_{x\to 2} x^3 - \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 1 \qquad \text{(by laws 1 and 2)}$$

$$= (\lim_{x\to 2} x)^3 - 4(\lim_{x\to 2} x)^2 + \lim_{x\to 2} 1 \qquad \text{(by laws 3 and 6)}$$

$$= 2^3 - 4(2)^2 + 1 = -7.$$

Ex. Evaluate $\lim_{x\to-3}(\frac{x^2-2x-6}{2-3x})$

$$\lim_{x\to-3}(\frac{x^2-2x-6}{2-3x}) = \frac{\lim_{x\to-3}(x^2-2x-6)}{\lim_{x\to-3}(2-3x)} \qquad \text{(by law 5)}$$

$$= \frac{\lim_{x\to-3} x^2 - \lim_{x\to-3} 2x - \lim_{x\to-3} 6}{\lim_{x\to-3} 2 - \lim_{x\to-3} 3x} \qquad \text{(by laws 1 and 2)}$$

$$= \frac{(\lim_{x\to-3} x)^2 - 2\lim_{x\to-3} x - \lim_{x\to-3} 6}{\lim_{x\to-3} 2 - 3\lim_{x\to-3} x} \qquad \text{(by laws 3 and 6)}$$

$$= \frac{(-3)^2-2(-3)-6}{2-3(-3)} = \frac{9+6-6}{2+9} = \frac{9}{11}.$$

Notice that for any polynomial or rational function (i.e., $\frac{p(x)}{q(x)}$; where $p(x), q(x)$ are polynomials) where a is in the domain of $f(x)$ we have:

$$\lim_{x\to a} f(x) = f(a).$$

That is, to evaluate the limit (in this case) you can just plug the value of a into the function.

Ex. Evaluate $\lim_{b\to 3}\left(\frac{\sqrt{2b^2-9}-2b+2}{4b-6}\right)$.

$$\lim_{b\to 3}\left(\frac{\sqrt{2b^2-9}-2b+2}{4b-6}\right)=\frac{\lim_{b\to 3}\left(\sqrt{2b^2-9}-2b+2\right)}{\lim_{b\to 3}(4b-6)}$$

(law 5)

$$=\frac{\lim_{b\to 3}\left(\sqrt{2b^2-9}\right)-\lim_{b\to 3} 2b+\lim_{b\to 3} 2}{\lim_{b\to 3} 4b-\lim_{b\to 3} 6}$$

(laws 1 and 2)

$$=\frac{\sqrt{\lim_{b\to 3}(2b^2-9)}-2\lim_{b\to 3} b+\lim_{b\to 3} 2}{4\lim_{b\to 3} b-\lim_{b\to 3} 6}$$

(laws 3 and 6)

$$=\frac{\sqrt{\lim_{b\to 3}(2b^2)-\lim_{b\to 3} 9}-2(3)+2}{4(3)-6}$$

(law 2)

$$=\frac{\sqrt{2\,(\lim_{b\to 3} b)^2-\lim_{b\to 3} 9}-4}{6}$$

(laws 3 and 6)

$$=\frac{\sqrt{2(3)^2-9}-4}{6}=\frac{\sqrt{9}-4}{6}=-\frac{1}{6}.$$

One-Sided Limits

Limit laws 1-6 also hold for one-sided limits. For example:

$$\lim_{x\to a^+}(f(x)g(x)) = (\lim_{x\to a^+} f(x))(\lim_{x\to a^+} g(x)).$$

However law #7 must be modified as follows. Assume $m, n > 0$ are integers.

$\lim_{x\to a^+}(f(x))^{\frac{n}{m}} = (\lim_{x\to a^+} f(x))^{\frac{n}{m}}$; provided $f(x) \geq 0$, for x near a with $x > a$, if m is even and n/m is reduced to lowest form.

$\lim_{x\to a^-}(f(x))^{\frac{n}{m}} = (\lim_{x\to a^-} f(x))^{\frac{n}{m}}$; provided $f(x) \geq 0$, for x near a with $x < a$, if m is even and n/m is reduced to lowest form.

Ex. Calculate $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1} f(x)$ if they exist if

$$f(x) = x\sqrt{x-1} \qquad if \quad 1 \leq x$$
$$= x\sqrt{2-x} \qquad if \ -1 \leq x < 1.$$

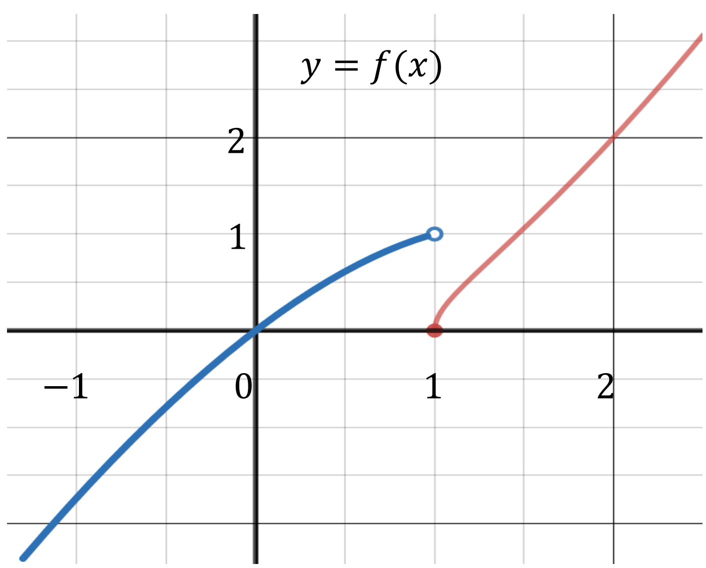
Start by sketching the graph of $f(x)$.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x\sqrt{x-1}\,) = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x\sqrt{2-x}\,) = 1$$

$\lim_{x\to 1} f(x) = DNE$, since

$$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x).$$



Indeterminate Forms

$\lim_{x\to a}(\frac{f(x)}{g(x)}) = 0$, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) \neq 0$, but exists.

$\lim_{x\to a}(\frac{f(x)}{g(x)}) =$ DNE, if $\lim_{x\to a} f(x) \neq 0$, but exists and $\lim_{x\to a} g(x) = 0$.

However, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}(\frac{f(x)}{g(x)})$ is called an **indeterminate form** and could equal any number or not exist, depending on the example. Two common techniques for evaluating indeterminate forms are factoring and multiplying by conjugates (when a square root is involved).

Ex. Evaluate $\lim_{x\to 2} \frac{x^2+x-6}{x^2-2x}$.

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-2x} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{(x+3)}{x} = \frac{5}{2}.$$

Ex. Evaluate $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$.

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{(9+6h+h^2)-9}{h}$$

$$= \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h\to 0}(6+h) = 6.$$

Ex. Evaluate $\lim_{t\to 0}\frac{\sqrt{t^2+16}-4}{t^2}$.

$$\lim_{t\to 0}\frac{\sqrt{t^2+16}-4}{t^2}=\lim_{t\to 0}\left(\frac{\sqrt{t^2+16}-4}{t^2}\right)\left(\frac{\sqrt{t^2+16}+4}{\sqrt{t^2+16}+4}\right)$$

$$=\lim_{t\to 0}\frac{t^2+16-16}{t^2(\sqrt{t^2+16}+4)}$$

$$=\lim_{t\to 0}\frac{t^2}{t^2(\sqrt{t^2+16}+4)}$$

$$=\lim_{t\to 0}\frac{1}{(\sqrt{t^2+16}+4)}=\frac{1}{8}\,.$$

Ex. Evaluate $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$.

$$\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}=\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}\left(\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}\right)$$

$$=\lim_{x\to 4}\frac{x+5-9}{(x-4)(\sqrt{x+5}+3)}$$

$$=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x+5}+3)}$$

$$=\lim_{x\to 4}\frac{1}{(\sqrt{x+5}+3)}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}\,.$$

The Squeeze Theorem: Assume the functions f, g, h satisfy

$f(x) \leq g(x) \leq h(x)$ for all values of x near $x = a$ except possibly at $x = a$. If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$. (Note: This theorem is still true if $a = \pm\infty$).

Ex. Sine and Cosine limits. It can be shown that for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$-|x| \leq sinx \leq |x| \qquad \text{and} \qquad 0 \leq 1 - cosx \leq |x|.$$

Using the Squeeze theorem show that:

a. $\lim_{x \to 0} sinx = 0$
b. $\lim_{x \to 0} cosx = 1.$

a. Using the first inequality, let $f(x) = -|x|$, $g(x) = sinx$, $h(x) = |x|$. Then $\lim_{x \to 0} f(x) = \lim_{x \to 0} -|x| = 0$, $\lim_{x \to 0} h(x) = \lim_{x \to 0} |x| = 0$, so by the squeeze theorem $\lim_{x \to 0} g(x) = \lim_{x \to 0} sinx = 0$.

b. Using the second inequality, let $f(x) = 0$, $g(x) = 1 - cosx$, $h(x) = |x|$.

Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} 0 = 0, \quad \lim_{x \to 0} h(x) = \lim_{x \to 0} |x| = 0,$$

so by the squeeze theorem $\lim_{x \to 0} g(x) = \lim_{x \to 0}(1 - cosx) = 0$.

Using our limit laws: $\lim_{x \to 0}(1 - cosx) = \lim_{x \to 0} 1 - \lim_{x \to 0} cosx = 0$,

So $\lim_{x \to 0} cosx = 1$.

Ex. Using the Squeeze theorem show that $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$.

Notice that for all real numbers t we have: $-1 \leq cost \leq 1$.

So for any $x \neq 0$ we have $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$.

Now multiply this inequality by x^2 (which we can do because $x^2 \geq 0$)

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2 .$$

Now let $f(x) = -x^2$, $g(x) = x^2 \cos\left(\frac{1}{x}\right)$, $h(x) = x^2$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} -x^2 = 0, \quad \lim_{x \to 0} h(x) = \lim_{x \to 0} x^2 = 0,$$

so by the squeeze theorem $\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$.