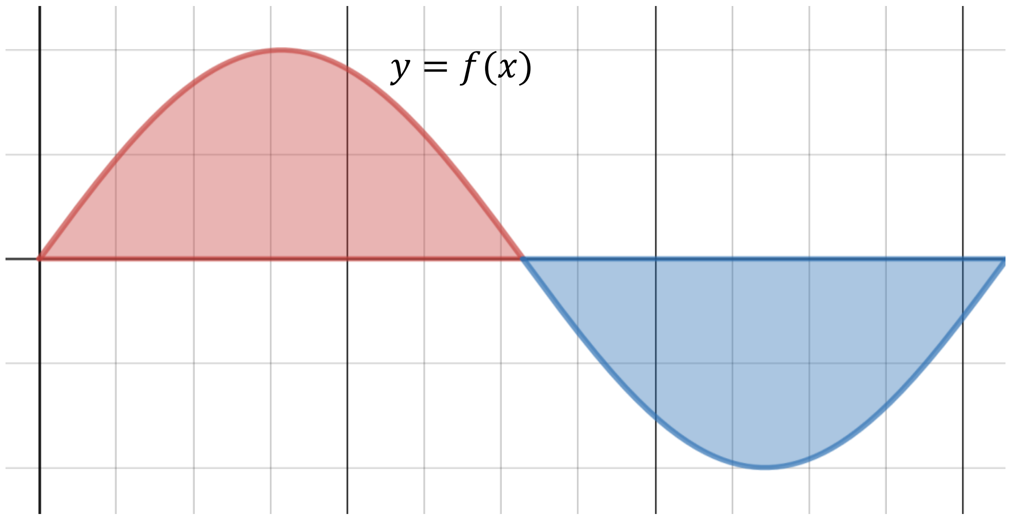
Area Between Curves

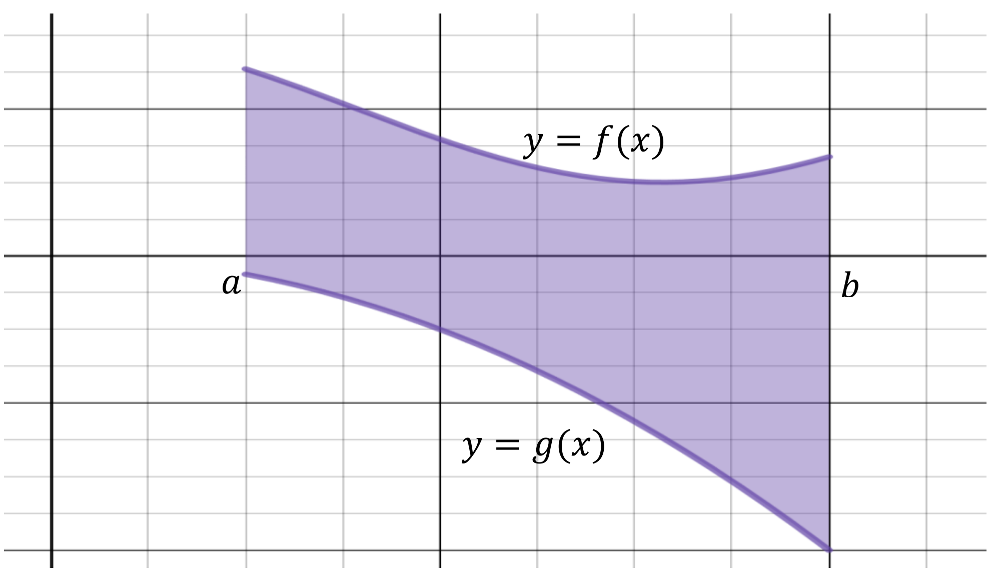
So far we have used a definite integral to find the area trapped between the graph of a function $f(x)$ and the x axis.

$$\text{Area} = \int_a^b |f(x)|dx$$



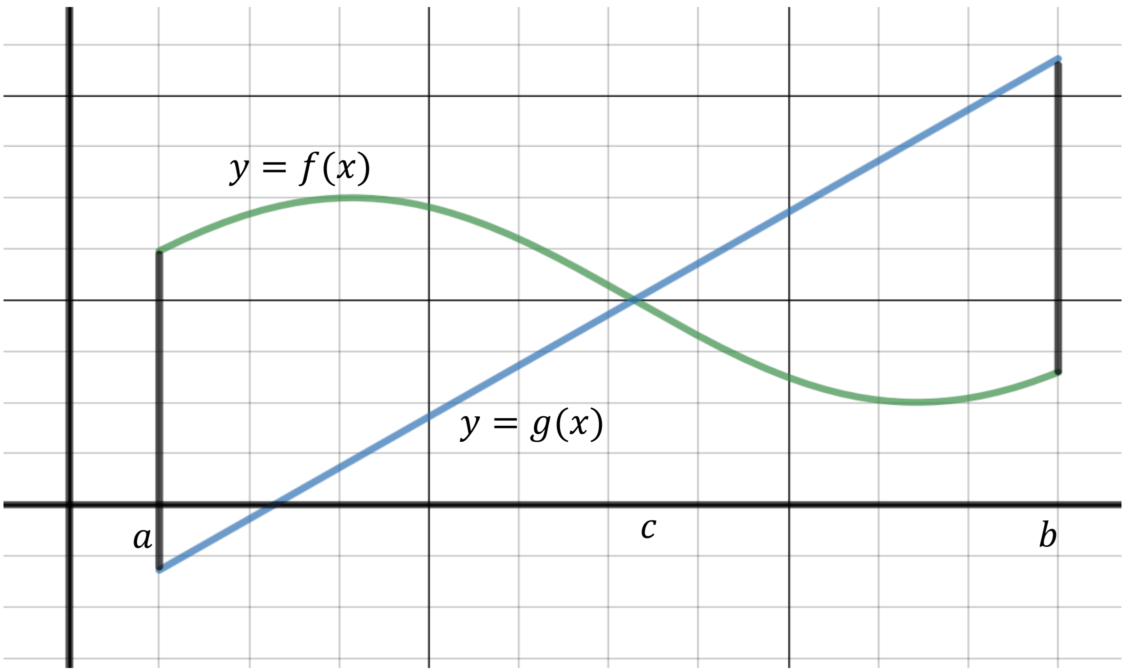
Def. Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$\textbf{Area} = \int_a^b \left(f(x) - g(x)\right)dx$$



To find the area bounded by 2 curves $\boldsymbol{y = f(x)}$ and $\boldsymbol{y = g(x)}$ we want to integrate the "top" curve minus the "bottom" curve. So in general, to find the area between 2 curves $y = f(x)$ and $y = g(x)$ (i.e., it may not be the case that $f(x) \geq g(x)$) on $[a, b]$):

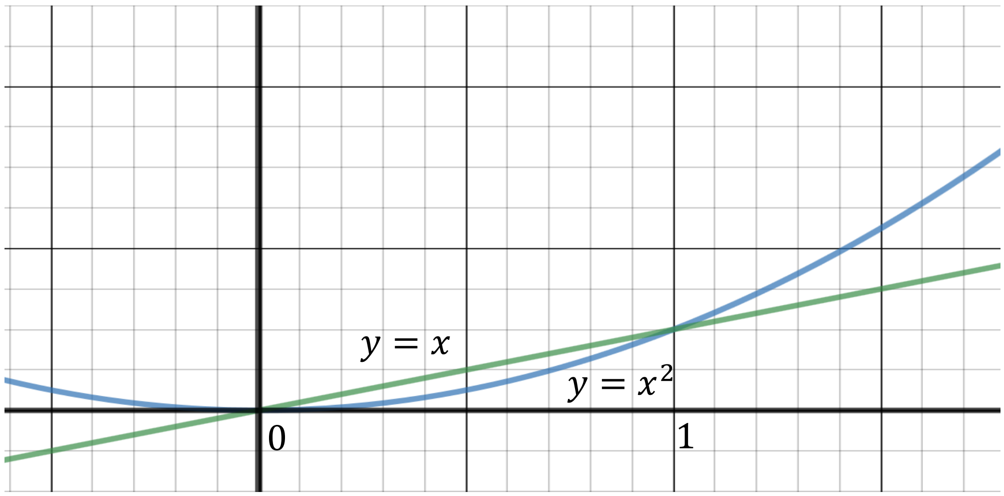
$$\textbf{Area}= \int_a^b |f(x) - g(x)|dx$$



$$\text{Area}= \int_a^c \big(f(x) - g(x)\big)dx + \int_c^b \big(g(x) - f(x)\big)dx$$

Ex. Find the area of the region bounded by the graphs of $y = x^2$ and $y = x$.

Start by sketching a graph of the two curves.



Next find their points of intersection.

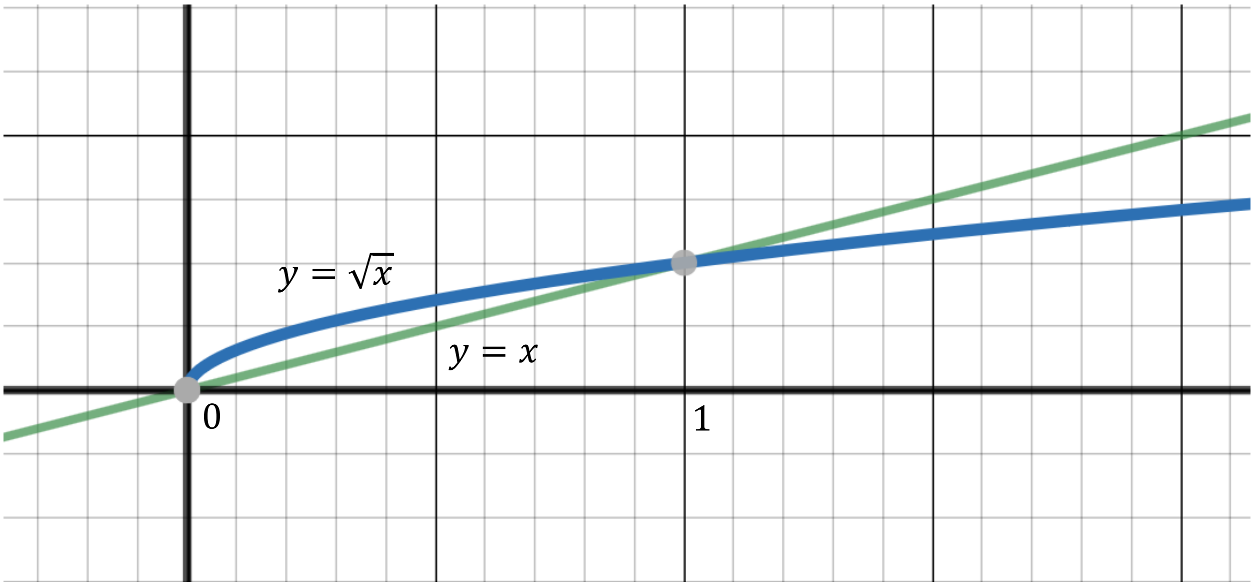
$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x-1) = 0 \implies \quad x = 0,1.$$

Area$= \int_0^1 (x - x^2)dx = (\frac{1}{2}x^2 - \frac{1}{3}x^3)\Big|_{x=0}^{x=1} = \left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0) = \frac{1}{6}$.

Ex. Find the area of the region bounded by the graphs of $y = \sqrt{x}$ and $y = x$. Start by sketching the curves (you need to know which is the top curve and which is the bottom curve).



Now find the intersection the two curves.

$$\sqrt{x} = x$$

$$x = x^2$$

$$0 = x^2 - x = x(x-1) \quad \Longrightarrow \quad x = 0,1.$$

Area$= \int_a^b (top\ curve - bottom\ curve)dx$

$$= \int_0^1 (\sqrt{x} - x)dx$$

$$= \int_0^1 \left(x^{\frac{1}{2}} - x\right) dx$$

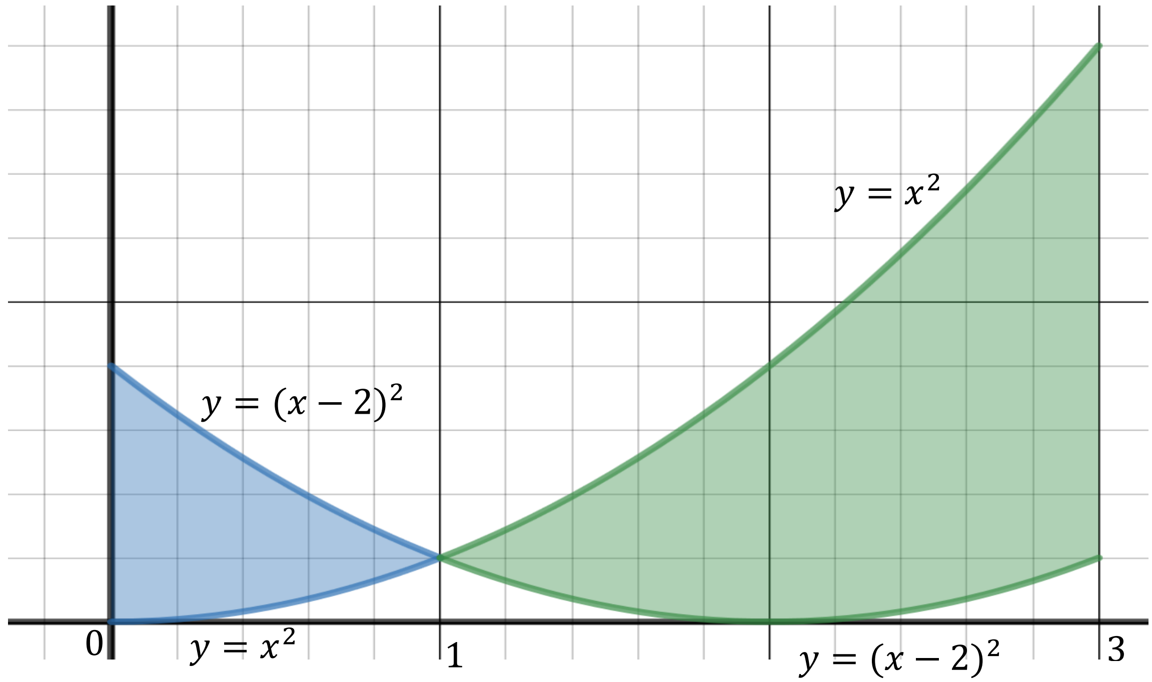
$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right)\Big|_{x=0}^{x=1}$$

$$= \left(\frac{2}{3}(1)^{\left(\frac{3}{2}\right)} - \frac{1}{2}(1)^2\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{2}(0)^2\right)$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}.$$

Ex. Find the area of the region bounded by the graphs of $y = x^2$, $y = (x-2)^2$, $x = 0$, $x = 3$.

Again, start by sketching the graphs.



We need to find the intersection of $y = x^2,\ y = (x-2)^2$ to determine for what values of x $\ y = x^2$ is the "top" curve and for what values of x $\ y = (x-2)^2$ is the top curve.

$x^2 = (x-2)^2$

$x^2 = x^2 - 4x + 4$

$0 = -4x + 4$; which means $4x = 4$ or $x = 1.$

So $y = (x-2)^2$ is the top curve for $0 \le x \le 1$ and $y = x^2$ is the top curve for $1 \le x \le 3$.

Area$= \int_0^1 ((x-2)^2 - x^2)\, dx + \int_1^3 (x^2 - (x-2)^2)\, dx$

$= \int_0^1 (x^2 - 4x + 4 - x^2) dx + \int_1^3 \left(x^2 - (x^2 - 4x + 4)\right) dx$

$= \int_0^1 (-4x + 4) dx + \int_1^3 (4x - 4) dx$

$$= (-2x^2 + 4x)\Big|_0^1 + (2x^2 - 4x)\Big|_1^3$$

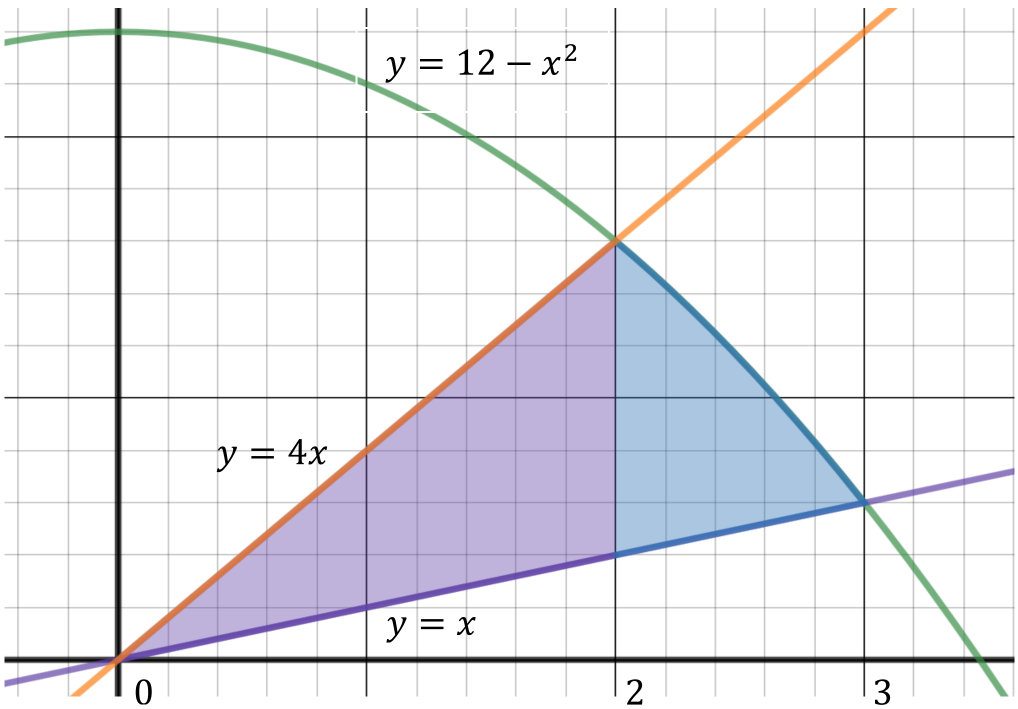
$$= [\big(-2(1)^2 + 4(1)\big) — \big(-2(0)^2 + 4(0)\big)]$$

$$+[\big(2(3)^2 - 4(3)\big) - \big(2(1)^2 - 4(1)\big)]$$

$$= (-2 + 4) - 0 + [(18 - 12) - (2 - 4)] = 2 + (6 + 2) = 10.$$

Ex. Find the area of the region in the first quadrant bounded by the graphs of $y = 12 - x^2$, $y = 4x$, and $y = x$.

Start by graphing each curve.



Points of intersection between $y = 12 - x^2$ and $y = 4x$:

$12 - x^2 = 4x$

$$0 = x^2 + 4x - 12$$

$$0 = (x + 6)(x - 2)$$

$x = -6, 2,$ but only $x = 2$ is in the first quadrant.

Points of intersection between $y = 12 - x^2$ and $y = x$:

$12 - x^2 = x$

$0 = x^2 + x - 12$

$0 = (x + 4)(x - 3)$

$x = -4, 3,$ but only $x = 3$ is in the first quadrant.

So $y = 4x$ is the top curve and $y = x$ is the bottom curve for $0 \leq x \leq 2$.

$y = 12 - x^2$ is the top curve and $y = x$ is the bottom curve for $2 \leq x \leq 3$.

Area$= \int_0^2 (4x - x)dx + \int_2^3 ((12 - x^2) - x)\, dx$

$$= \int_0^2 3x dx + \int_2^3 (12 - x^2 - x)dx$$

$$= \left(\tfrac{3}{2}x^2\right)\Big|_0^2 + \left(12x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big|_2^3$$

$$= \left(\tfrac{3}{2}(2^2) - \tfrac{3}{2}(0)^2\right) + [\left(12(3) - \tfrac{1}{3}(3^3) - \tfrac{1}{2}(3^2)\right)$$
$$-(12(2) - \tfrac{1}{3}(2^3) - \tfrac{1}{2}(2^2))]$$

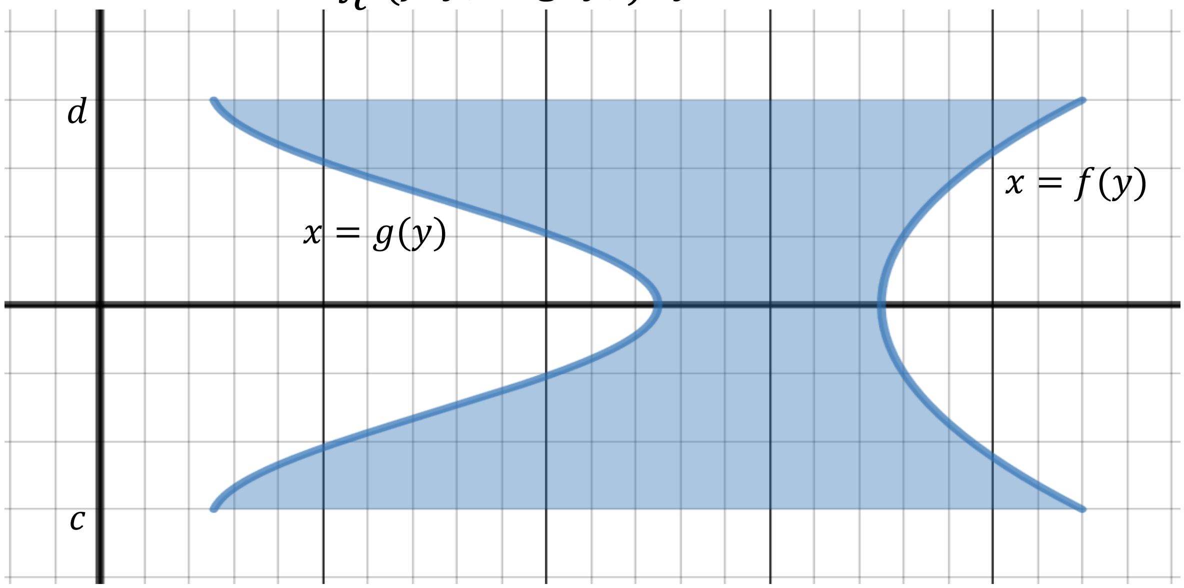
$$= (6 - 0) + [\left(36 - 9 - \tfrac{9}{2}\right) - \left(24 - \tfrac{8}{3} - 2\right)]$$

$$= 6 + \frac{45}{2} - \left(\frac{58}{3}\right) = 6 + \frac{135}{6} - \frac{116}{6} = \frac{55}{6}.$$

Area of a Region between $x = f(y)$ and $x = g(y)$

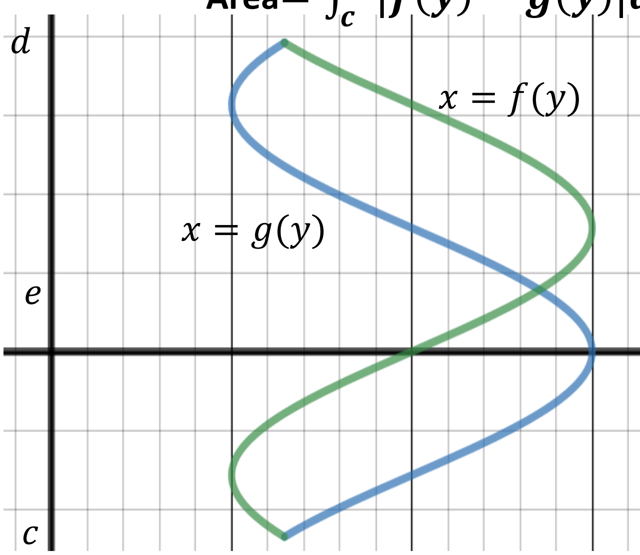
Def. Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on an interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$\textbf{Area} = \int_c^d \big(f(y) - g(y)\big)\,dy$$



To find the area bounded by any 2 continuous curves $x = f(y)$ and $x = g(y)$ we want to integrate the curve "furthest to the right" minus the curve "furthest to the left". This is equivalent to:

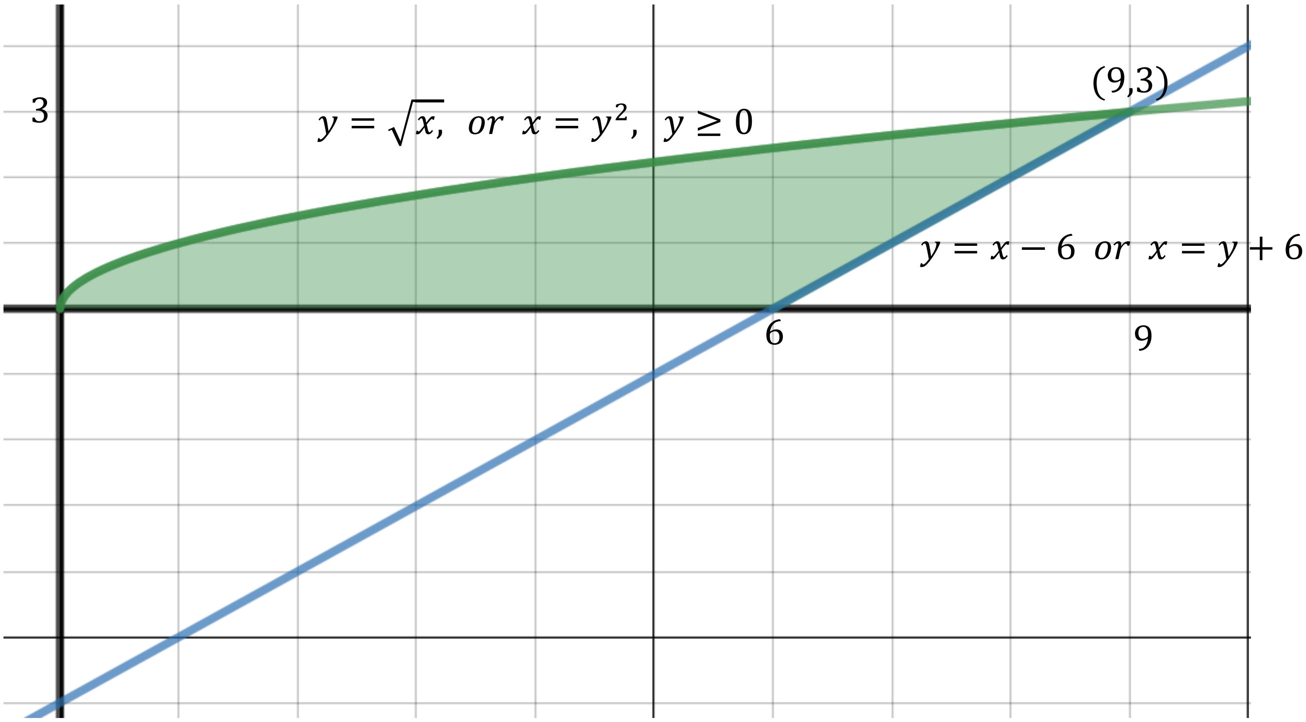
$$\textbf{Area} = \int_c^d |f(y) - g(y)|\,dy.$$



$$\text{Area} = \int_c^e \big(g(y) - f(y)\big)\,dy + \int_e^d \big(f(y) - g(y)\big)\,dy$$

Ex. Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 6$.

Start by sketching the graphs.



Notice that this problem can be done as 2 curves $y = f(x)$ and $y = g(x)$ OR $x = f(y)$ and $x = g(y)$. It's easier the second way because we can do it with one integral instead of 2.

$y = \sqrt{x}$ is the same as $x = y^2$ in the first quadrant, $y = x - 6$ is the same as $x = y + 6$.

Find the intersection in the first quadrant of $x = y^2$ and $x = y + 6$.

$$y^2 = y + 6$$

$$y^2 - y - 6 = 0$$

$$(y - 3)(y + 2) = 0$$

$y = 3, -2$ but only $y = 3$ is in the first quadrant.

So $x = y + 6$ is the curve farthest to the right between $y = 0$ and $y = 3$.

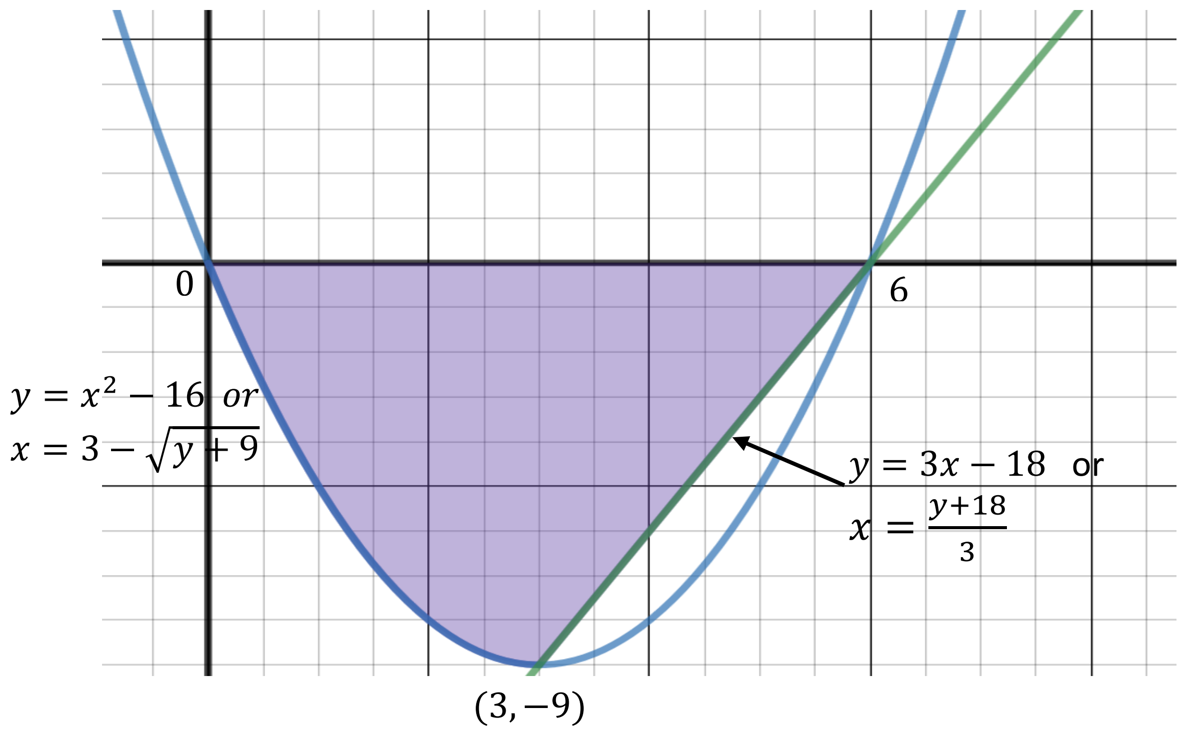
Area$= \int_0^3 ((y+6) - y^2)dy = \int_0^3 (-y^2 + y + 6)dy$

$$= -\frac{1}{3}y^3 + \frac{1}{2}y^2 + 6y \Big|_0^3$$

$$= \left(-\frac{1}{3}(3^3) + \frac{1}{2}(3^2) + 6(3)\right) - \left(-\frac{1}{3}(0^3) + \frac{1}{2}(0^2) + 6(0)\right)$$

$$= \left(-9 + \frac{9}{2} + 18\right) - 0 = \frac{27}{2}.$$

Ex. Write down (but don't evaluate) integrals that represent the area bounded by the curves $y = x^2 - 6x$, $y = 3x - 18$, and $y = 0$ (i.e. the x axis) first in terms of x, and then in terms of y. Start by drawing the region.



Find the points of intersection between $y = x^2 - 6x$, $y = 3x - 18$ and $y = 0$:

$$x^2 - 6x = 3x - 18$$

$$x^2 - 9x + 18 = 0$$

$$(x-3)(x-6) = 0 \quad \Longrightarrow x = 3, 6.$$

$(3, -9)$, $(6,0)$ are points of intersection between $y = x^2 - 6x$ and $y = 3x - 18$.

$y = x^2 - 6x$ and $y = 0$ intersect when $x^2 - 6x = 0 \Longrightarrow x = 0, 6$.

$(0,0), (6,0)$ are points of intersection.

$y = 3x - 18$ and $y = 0$ intersect when $3x - 18 = 0 \quad \Longrightarrow x = 6$.

$(6,0)$ is the point of intersection.

In terms of x: Between $x = 0$ and $x = 3$, the top curve is $y = 0$ and the bottom curve is $y = x^2 - 6x$. Between $x = 3$ and $x = 6$ the top curve is $y = 0$ and the bottom curve is $y = 3x - 18$.

Area of region$= \int_0^3 (0 - (x^2 - 6x))\, dx + \int_3^6 (0 - (3x - 18)\, dx$

$$= \int_0^3 (-x^2 + 6x) dx + \int_3^6 (-3x + 18) dx.$$

In terms of y: First we need the curve written $x = f(y)$ and $x = g(y)$. Let's start with: $y = 3x - 18$. Solving for x in terms of y we get:

$$y + 18 = 3x \quad \Longrightarrow \quad \frac{y+18}{3} = x.$$

Now for $y = x^2 - 6x$ we need to complete the square first.

$$y = (x^2 - 6x + 9) - 9$$

$$y = (x - 3)^2 - 9$$

$$y + 9 = (x - 3)^2$$

$$\pm\sqrt{y + 9} = x - 3$$

$$3 \pm \sqrt{y + 9} = x\ .$$

But from the graph we can see that x is between 0 and 3, so

$$3 - \sqrt{y + 9} = x\ .$$

The curve furthest to the right is: $x = \frac{y+18}{3}$

The curve furthest to the left is: $x = 3 - \sqrt{y + 9}$

Area of region$=\int_{y=-9}^{y=0} [\frac{y+18}{3} - (3 - \sqrt{y + 9})]\ dy$.