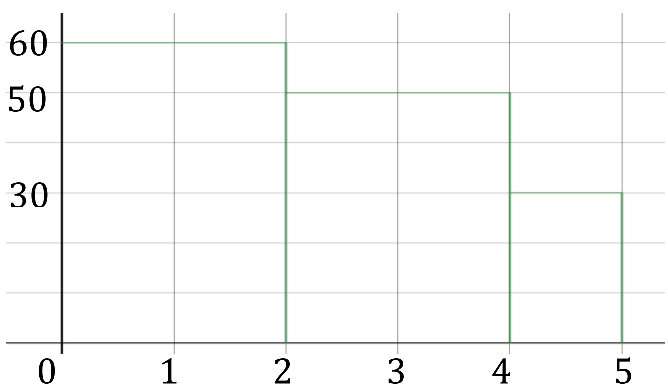
Approximating the Area under a Curve

Let's start with the area under a velocity curve.

Notice if $v(t) = 60$, then the area under the curve between $t = t_1$ and $t = t_2$ is just the displacement (if the velocity is positive then the displacement equals the distance, if it's negative then the displacement equals the negative of the distance).

$$\text{If } v(t) = \begin{cases} 60 & 0 \le t < 2 \\ 50 & 2 \le t < 4 \\ 30 & 4 \le t \le 5 \end{cases}$$



The area underneath the graph of $v(t)$ for $0 \le t \le 5$ is the displacement for $0 \le t \le 5$.

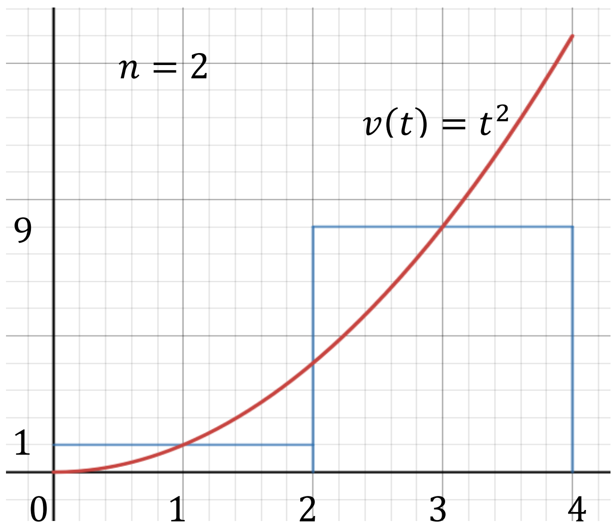
Now suppose we let $v(t) = t^2$ for $0 \le t \le 4$. We can approximate the displacement (i.e. the area underneath the graph of $v(t)$) by breaking the interval $[0,4]$ in to subintervals of equal length and approximating the velocity by a constant velocity.

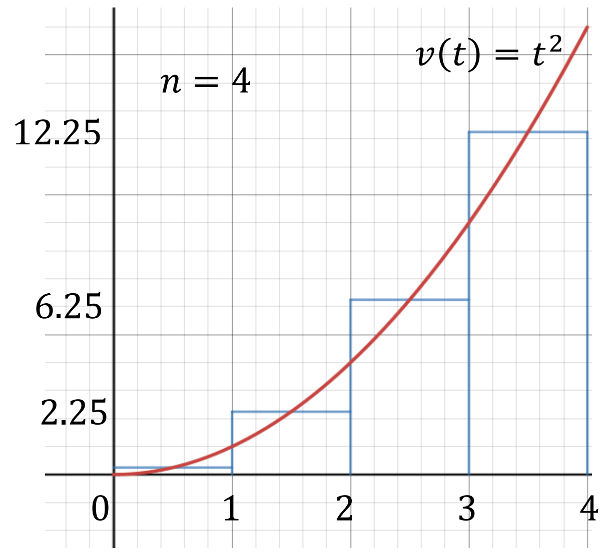
Ex. Approximate the area under the curve $v(t) = t^2$ for $0 \leq t \leq 8$ by assuming that the velocity at the midpoint was the constant velocity over the interval.

a. Let $n = 2, \quad [0,2], \ [2,4]$

b. Let $n = 4, \quad [0,1], \ [1,2], \ [2,3], \ [3,4]$

c. Let $n = 8, \quad [0,.5], \ [.5,1], \ [1,1.5], \dots, \ [3.5,4]$





a. Area$\approx v(1)\cdot 2 + v(3)\cdot 2 = 1^2(2) + 3^2(2) = 2 + 18 = 20$

b. Area$\approx v\left(\frac{1}{2}\right)\cdot(1) + v\left(\frac{3}{2}\right)\cdot 1 + v\left(\frac{5}{2}\right)\cdot 1 + v\left(\frac{7}{2}\right)\cdot 1$

$= (\frac{1}{2})^2(1) + (\frac{3}{2})^2(1) + (\frac{5}{2})^2(1) + (\frac{7}{2})^2(1)$

$= .25 + 2.25 + 6.25 + 12.25 = 21.$

c. Area$\approx v(.25)\cdot(.5) + v(.75)\cdot(.5) + \cdots + v(3.75)\cdot(.5)$

$= 21.25.$

Number of Subintervals	Length of each subinterval	Approx. displacement (area under curve)
1	4	16
2	2	20
4	1	21
8	0.5	21.25
16	0.25	21.3125
32	0.125	21.328125

The approximate areas are converging to the area underneath the curve (which is actually $\frac{64}{3} = 21.\overline{3}$).

Def. Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0,\ x_1],[x_1,\ x_2],[x_2,\ x_3],\dots,\ [x_{n-1},\ x_n]$$

of equal length $\Delta x = \frac{b-a}{n}$, with $x_0 = a$ and $x_n = b$. The endpoints $x_0, x_1, x_2, \dots, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general the kth grid point is

$$x_k = a + k\Delta x, \text{ for } k = 0,1,2,3,\dots,n.$$

For the purposes of approximating the area under the curve with rectangles, we don't have to take the value of the function at the midpoint of the interval as the height of the rectangle. We can actually take the value of the function at any point in the interval. If we call ${x_k}^*$ any point in the kth interval then the sum of the areas of the rectangles (assuming $f(x) \geq 0$) is given by the following **Riemann Sum**

$$\mathrm{f}({x_1}^*)\Delta x + \mathrm{f}({x_2}^*)\Delta x + \cdots + \mathrm{f}({x_n}^*)\Delta x$$

for f on $[a, b]$. The sum is called

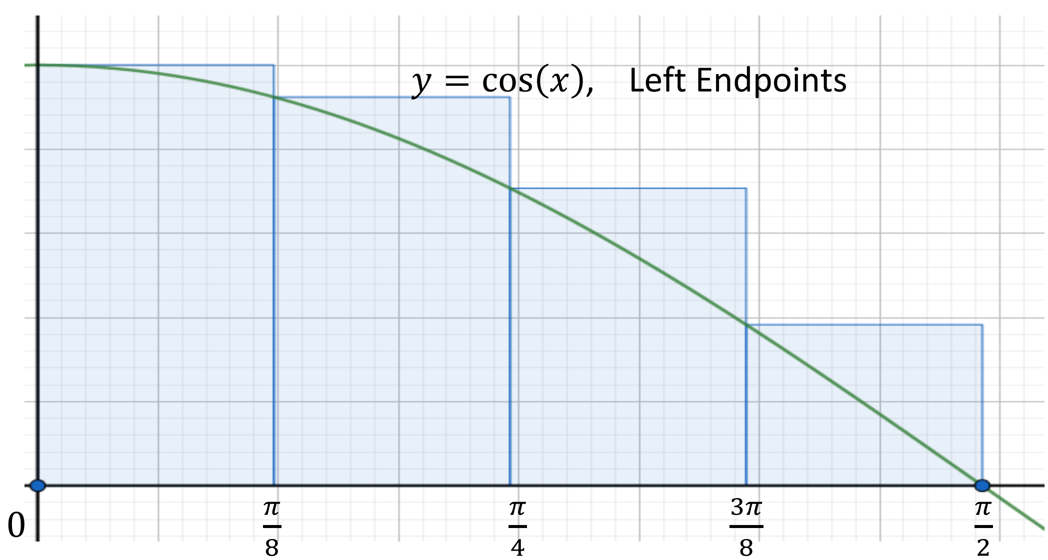
1. A **left Riemann sum** if ${x_k}^*$ is the left endpoint of $[x_{k-1,} x_k]$
2. A **right Riemann sum** if ${x_k}^*$ is the right endpoint of $[x_{k-1,} x_k]$
3. A **midpoint Riemann sum** if ${x_k}^*$ is the midpoint of $[x_{k-1,} x_k]$

$k = 1,2, \ldots, n.$

Ex. Let R be the region bounded by $f(x) = cosx$ for x between 0 and $\frac{\pi}{2}$ and the x-axis. Approximate the area of R using a left Riemann sum and a right Riemann sum with $n = 4$ (4 subdivisions).

$$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}.$$

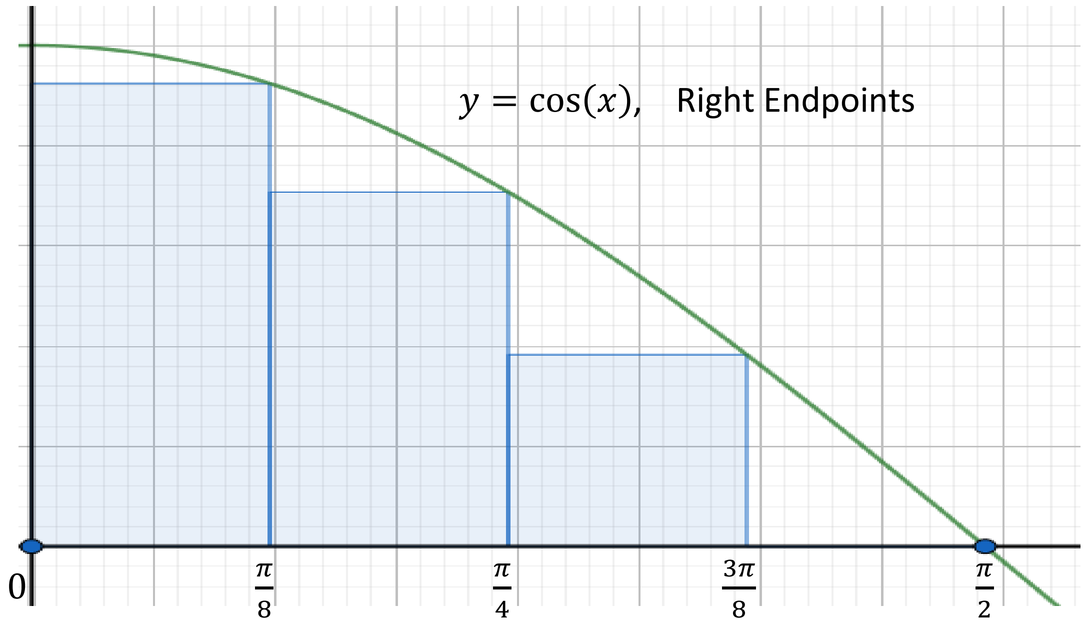
$$x_0 = 0, \quad x_1 = \frac{\pi}{8}, \quad x_2 = \frac{\pi}{4}, \quad x_3 = \frac{3\pi}{8}, \quad x_4 = \frac{\pi}{2}.$$



Left Riemann sum$= f(0)\Delta x + f\left(\frac{\pi}{8}\right)\Delta x + f\left(\frac{\pi}{4}\right)\Delta x + f(\frac{3\pi}{8})\Delta x$

$$= (\cos 0)\left(\frac{\pi}{8}\right) + (\cos\frac{\pi}{8})\left(\frac{\pi}{8}\right) + (\cos\frac{\pi}{4})\left(\frac{\pi}{8}\right) + (\cos\frac{3\pi}{8})(\frac{\pi}{8})$$

$$\approx 1.183.$$



Right Riemann sum$= f\left(\frac{\pi}{8}\right)\Delta x + f\left(\frac{\pi}{4}\right)\Delta x + f\left(\frac{3\pi}{8}\right)\Delta x + f(\frac{\pi}{2})\Delta x$

$$= (\cos\frac{\pi}{8})\left(\frac{\pi}{8}\right) + (\cos\frac{\pi}{4})\left(\frac{\pi}{8}\right) + \left(\cos\frac{3\pi}{8}\right)\left(\frac{\pi}{8}\right) + (\cos\frac{\pi}{2})(\frac{\pi}{8})$$

$$\approx 0.791.$$

In this example, we would expect the right Riemann sum to be less than the left Riemann sum because the function is decreasing for $0 \leq x \leq \frac{\pi}{2}$.

Ex. Find the midpoint Riemann sum for the previous example.

Midpoint Riemann sum$= f(\frac{\pi}{16})\Delta x + f\left(\frac{3\pi}{16}\right)\Delta x + f\left(\frac{5\pi}{16}\right)\Delta x + f(\frac{7\pi}{16})\Delta x$

$$= (\cos\frac{\pi}{16})\left(\frac{\pi}{8}\right) + (\cos\frac{3\pi}{16})\left(\frac{\pi}{8}\right) + (\cos\frac{5\pi}{16})\left(\frac{\pi}{8}\right) + (\cos\frac{7\pi}{16})(\frac{\pi}{8})$$

$$\approx 1.006.$$

Sigma (Summation) notation:

$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$

$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

$\sum_{k=2}^{4}(k^2 + k) = (2^2 + 2) + (3^2 + 3) + (4^2 + 4) = 6 + 12 + 20 = 38$

Constant Multiple Rule: $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

Addition Rule: $\sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j$.

Useful sums of powers of integers

$\sum_{k=1}^{n} c = cn$

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

We can now write Riemann sums in sigma notation.

$\mathrm{f}(x_1{}^*)\Delta x + \mathrm{f}(x_2{}^*)\Delta x + \cdots + \mathrm{f}(x_n{}^*)\Delta x = \sum_{k=1}^{n} \mathrm{f}(x_k{}^*)\Delta x$.

For left Riemann sums $x_k{}^* = a + (k-1)\Delta x$

For right Riemann sums $x_k{}^* = a + (k)\Delta x$

For midpoint Riemann sums $x_k{}^* = a + (k - \frac{1}{2})\Delta x$

It's much easier to work with Riemann sums using sigma notation.

Ex. Calculate the right Riemann sum for $f(x) = x^3 + 1$ between $a = 0$ and $b = 2$ using $n = 50$ subintervals.

$\Delta x = \frac{b-a}{n} = \frac{2-0}{50} = 0.04$

Right Riemann sum: $\sum_{k=1}^{n} \mathrm{f}(x_k{}^*)\Delta x, \quad x_k{}^* = a + (k)\Delta x = 0 + (k)0.04$

$$\sum_{k=1}^{n} \mathrm{f}(x_k{}^*)\Delta x = \sum_{k=1}^{50}[(k)^3(0.04)^3 + 1](0.04)$$
$$= (0.04)^4 \sum_{k=1}^{50}(k)^3 + (0.04)\sum_{k=1}^{50} 1$$

Now using the first and fourth sum formulas on the previous page:

$$= (0.04)^4 \left(\frac{50^2(51)^2}{4}\right) + (0.04)(50)$$
$$= 6.1616.$$