Antiderivatives

Def. A function F is an **antiderivative** of f on an interval I provided
$F'(x) = f(x)$, for all x in I.

Antiderivatives will become very important when we talk about integration.

If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, i.e. $F'(x) = f(x)$ and $G'(x) = f(x)$, Then $G'(x) = F'(x)$. As we saw from the Mean Value Theorem, this means that $G(x) = F(x) + C$, where C is a constant.

So an antiderivative of a function is not unique, but any antiderivative of a function f differs from any other antiderivative of f by a constant.

So far we have developed a number of formulas for derivatives of functions. To find an antiderivative of a function $f(x)$, we need to go "backwards". That is we need to find a function $F(x)$ such that $F'(x) = f(x)$.

Ex. Find all antiderivatives of $f(x) = 1$.

This one isn't so bad. We have to find a function $F(x)$ such that $F'(x) = 1$. If we take $F(x) = x$, that will work. So $F(x) = x$ is an antiderivative of $f(x) = 1$. If we want all antiderivatives of $f(x) = 1$, we take $F(x) = x + C$.

Ex. Find all antiderivatives of $f(x) = x$.

So we need a function $F(x)$ such that $F'(x) = x$. If we take $F(x) = \frac{1}{2}x^2$ we will have an antiderivative of $f(x) = x$. To get all antiderivatives we take $F(x) = \frac{1}{2}x^2 + C$.

Ex. Find all antiderivatives of $h(x) = x^n$, where n is a positive integer.

We know when we take a derivative of x^n, n a positive integer, we get $\frac{d}{dx}(x^n) = nx^{n-1}$.

So if we want to go "backwards" let's try taking as an antiderivative of x^n, $H(x) = \frac{1}{n+1}x^{n+1}$.

Notice that if n is a positive integer (or even a positive rational number) that

$H'(x) = x^n$.

So all of the antiderivatives of $h(x) = x^n, n$ a positive integer, are given by $H(x) = \frac{1}{n+1}x^{n+1} + C$.

In particular, when $n = 3$, all of the antiderivatives of $g(x) = x^3$ are given by $G(x) = \frac{1}{4}x^4 + C$.

When $n = 2$, all of the antiderivatives of $f(x) = x^2$ are given by $F(x) = \frac{1}{3}x^3 + C$.

Notice that if $F(x)$ is an antiderivative of $f(x)$, then $kF(x)$ is an antiderivative of $kf(x)$, where k is a constant.

Ex. Find all of the antiderivatives of $f(x) = -9x^5$ and $g(x) = \frac{x^8}{3}$.

We know that an antiderivative of x^5 is given by $\frac{1}{6}x^6$ so an antiderivative of $-9x^5$ is given by $-9\left(\frac{1}{6}x^6\right) = -\frac{3}{2}x^6$. Thus all of the antiderivatives of $f(x) = -9x^5$ are given by $F(x) = -\frac{3}{2}x^6 + C$.

We know that an antiderivative of x^8 is given by $\frac{1}{9}x^9$ so an antiderivative of $\frac{x^8}{3} = \frac{1}{3}x^8$ is given by $\frac{1}{3}\left(\frac{1}{9}x^9\right) = \frac{1}{27}x^9$. Thus all of the antiderivatives of $g(x) = \frac{x^8}{3}$ are given by $G(x) = \frac{1}{27}x^9 + C$.

The symbol we use for the antiderivatives of a function $f(x)$ is what's called the **indefinite integral of $\boldsymbol{f(x)}$, written $\boldsymbol{\int f(x)dx}$.**

So we would say $\int -9x^5 dx = -\frac{3}{2}x^6 + C$

Power Rule for Indefinite Integrals

$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ where $n \neq -1$ is a real number and C is any constant.

This follows from the fact that $\frac{d}{dx}\left(\frac{1}{n+1} x^{n+1} + C\right) = \frac{d}{dx}\left(\frac{1}{n+1} x^{n+1}\right) + \frac{d}{dx}(C)$

$= x^n.$

Note: I'm "cheating" a bit here since we only know the derivative power rule $\frac{d}{dx}(x^n) = nx^{n-1}$ when n is a rational number. However, this rule as well as the Power Rule for indefinite integrals is true for n a real number.

Just as in the case with taking derivatives, the indefinite integral of the sum or difference of 2 functions is the sum or difference of the indefinite integrals. The indefinite integral of a constant times a function is that constant times the indefinite integral of the function.

Constant Multiple and Sum/Difference Rules

$\int cf(x)dx = c \int f(x)dx$

$\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx.$

Ex. Find the following indefinite integrals (Try to first write the function you are integrating, called the integrand, as a sum of terms that are just constants times powers of the variable)

a. $\int(2x^3 - 4 + \frac{6}{\sqrt{x}})dx$
b. $\int\left(\frac{2x^{11}-3x^{-5}}{x^5}\right)dx$
c. $\int(y^2 + 3)(y^3 - 1)dy$

a. $\int(2x^3 - 4 + \frac{6}{\sqrt{x}})dx = \int(2x^3 - 4 + 6x^{-\frac{1}{2}})dx$

$$= 2\int x^3 dx - 4\int 1\,dx + 6\int x^{-\frac{1}{2}}dx$$

$$= 2\left(\frac{1}{4}x^4\right) - 4x + 6\left(\frac{1}{\frac{1}{2}}x^{\frac{1}{2}}\right) + C$$

$$= \frac{1}{2}x^4 - 4x + 6\left(2x^{\frac{1}{2}}\right) + C$$

$$= \frac{1}{2}x^4 - 4x + 12\left(x^{\frac{1}{2}}\right) + C$$

b. $\int\left(\frac{2x^{11}-3x^{-5}}{x^5}\right)dx = \int(2\frac{x^{11}}{x^5} - 3\frac{x^{-5}}{x^5})dx$

$$= \int(2x^6 - 3x^{-10})\,dx$$

$$= 2\int x^6 dx - 3\int x^{-10}dx$$

$$= 2\left(\frac{1}{7}x^7\right) - 3\left(\frac{1}{-9}\right)x^{-9} + C$$

$$= \frac{2}{7}x^7 + \frac{1}{3}x^{-9} + C$$

c. $\int(y^2 + 3)(y^3 - 1)dy = \int(y^5 - y^2 + 3y^3 - 3)dy$

$$= \int y^5 dy - \int y^2 dy + 3\int y^3 dy - 3\int 1dy$$

$$= \frac{1}{6}y^6 - \frac{1}{3}y^3 + 3\left(\frac{1}{4}y^4\right) - 3y + C$$

$$= \frac{1}{6}y^6 + \frac{3}{4}y^4 - \frac{1}{3}y^3 - 3y + C.$$

Indefinite Integrals of Trig Functions:

A Table of derivatives can help us develop a table of antiderivatives

1. $\frac{d}{dx}(sinax) = acosax$ $\quad\quad$ $\int cosax\, dx = \frac{1}{a}\sin(ax) + C$
2. $\frac{d}{dx}(cosax) = -asinax$ $\quad\quad$ $\int sinax\, dx = -\frac{1}{a}\cos(ax) + C$
3. $\frac{d}{dx}(tanax) = asec^2ax$ $\quad\quad$ $\int sec^2ax\, dx = \frac{1}{a}\tan(ax) + C$
4. $\frac{d}{dx}(cotax) = -acsc^2ax$ $\quad\quad$ $\int csc^2ax\, dx = -\frac{1}{a}\cot(ax) + C$
5. $\frac{d}{dx}(secax) = a(secax)(tanax)$ $\quad\quad$ $\int (secax)(tanax)\, dx = \frac{1}{a}\sec(ax) + C$
6. $\frac{d}{dx}(cscax) = -a(cscax)(cotax)$ $\quad\quad$ $\int (cscax)(cotax)dx = -\frac{1}{a}\csc(ax) + C$

Ex. Evaluate the following indefinite integrals

a. $\int \cos(6x)\, dx$
b. $\int csc^2(3x)\ \ dx$
c. $\int (2\sin(4x) + 6sec^2(\frac{x}{3}))dx$

a. $\int \cos(6x)\, dx = \frac{1}{6}\sin(6x) + C$

b. $\int csc^2(3x)\ \ dx = -\frac{1}{3}\cot(3\text{x}) + \text{C}$

c. $\int \left(2\sin(4x) + 6sec^2\left(\frac{x}{3}\right)\right) dx$

$$= 2\int \sin(4x)\, dx + 6\int sec^2\left(\frac{x}{3}\right) dx$$

$$= 2(-\frac{1}{4}\cos(4x)) + 6\left(3\tan\left(\frac{x}{3}\right)\right) + C$$

$$= -\frac{1}{2}\cos(4x) + 18\tan\left(\frac{x}{3}\right) + C.$$

Ex. Evaluate $\int[3\sec(x)\tan(x) - 2\csc(5x)\cot(5x)]dx$

$$\int[3\sec(x)\tan(x) - 2\csc(5x)\cot(5x)]dx$$
$$= 3\int\sec(x)\tan(x)\,dx - 2\int\csc(5x)\cot(5x)\,dx$$
$$= 3\sec(x) + 2\left(\frac{1}{5}\right)\csc(5x) + C$$
$$= 3\sec(x) + \frac{2}{5}\csc(5x) + C.$$

<u>Differential Equations</u>

A **Differential Equation** is an equation that involves the derivative (or derivatives) of a function as well as possibly the function itself. A solution to a differential equation is a function that satisfies the equation. We have already solved some elementary differential equations when we found antiderivatives.

Ex. Solve the following differential equation

$$f'(x) = 3x^2 + 7.$$

So in this case we are looking for all the antiderivatives of $3x^2 + 7$. In other words we want $\int(3x^2 + 7)dx = x^3 + 7x + C$.

So any function of the form $f(x) = x^3 + 7x + C$ is a solution to this differential equation.

In many cases of differential equations, they are accompanied by an **Initial Condition**. An initial condition is a condition on the unknown function f, that will allow us to calculate what the constant term C is.

Ex. Solve the following differential equation

$$f'(x) = 3x^2 + 7$$

with the initial condition that $f(1) = 4$.

In the previous example we found the **general solution** to this differential equation: $f(x) = x^3 + 7x + C$.

Now we know that $f(1) = 4$, so we can't use just any value of the constant C.

$$4 = f(1) = 1^3 + 7(1) + C = 8 + C \quad \Longrightarrow \quad C = -4.$$

So $C = -4$ and the solution to this **Initial Value Problem** (i.e. a differential equation together with an initial condition(s)) is

$$f(x) = x^3 + 7x - 4.$$

Ex. Solve the differential equation: $f'(x) = 9\sin(3x) + 8x^3$ with $f(0) = 5$.

$$f(x) = \int (9\sin(3x) + 8x^3)dx$$

$$= 9\int \sin(3x)\,dx + 8\int x^3 dx$$

$$= 9\left(-\tfrac{1}{3}\cos(3x)\right) + 8\left(\tfrac{1}{4}x^4\right) + C \quad = -3\cos(3x) + 2x^4 + C.$$

$f(0) = 5$; so

$$5 = f(0) = -3\cos(0) + 2(0)^4 + C$$

$$5 = -3 + C \quad \Longrightarrow \quad C = 8.$$

$$f(x) = -3\cos(3x) + 2x^4 + 8.$$

<u>Linear Motion</u>

Initial Value Problems for Velocity and Position.

Suppose an object moves in a line with velocity $v(t), t \geq 0$. Then its position can be found by solving the initial value problem

$$s'(t) = v(t), \;\; s(0) = s_0, \text{ where } s_0 \text{ is known.}$$

If the acceleration of the object $a(t)$ is given, then the velocity can be found by solving the initial value problem

$$v'(t) = a(t), \;\; v(0) = v_0, \text{ where } v_0 \text{ is known.}$$

Ex. Runner A begins at point $s(0) = 0ft$ with a velocity of $v(t) = (2t + 1)ft/sec$. Runner B begins with a head start at the point $S(0) = 12ft$ with a velocity of $V(t) = 5ft/sec$. Find the positions of the runners for $t \geq 0$ and determine when runner A overtakes runner B.

For runner A: $s'(t) = v(t) = 2t + 1, \;\; s(0) = 0.$

So $s(t) = \int (2t + 1)dt = t^2 + t + C$

$0 = s(0) = 0^2 + 0 + C,$ thus $C = 0.$

So $s(t) = t^2 + t.$

For runner B: $S(t) = V(t) = 5, \; S(0) = 12$

$S(t) = \int 5dt = 5t + C$

$12 = S(0) = 5(0) + C,$ thus $C = 12$

So $S(t) = 5t + 12.$

To find when Runner A overtakes Runner B solve $t^2 + t = 5t + 12$.

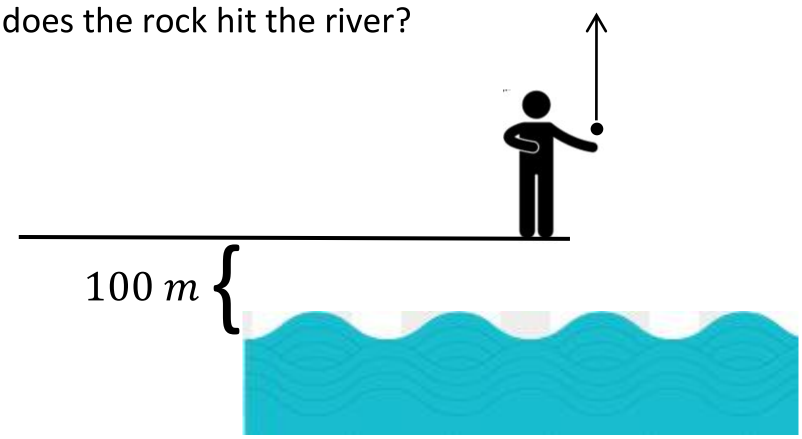
$$t^2 - 4t - 12 = 0$$

$$(t-6)(t+2) = 0$$

$$t = 6, -2 \text{ since } t \geq 0, \quad t = 6sec.$$

Ex. Neglecting air resistance the Earth's gravitational pull accelerates objects down toward the Earth at a constant acceleration of $a(t) = -9.8m/s^2$. A rock is thrown vertically upward at $t = 0$ with a velocity of $30m/s$ from the edge of a cliff that is $100m$ above a river.

a. Find the velocity $v(t), t \geq 0$.
b. Find the position $s(t),\ t \geq 0$.
c. Find the maximum height of the rock.
d. At what speed does the rock hit the river?



a. $v'(t) = a(t) = -9.8, \ v(0) = 30$

$$v(t) = \int -9.8dt = -9.8t + C$$

$$30 = v(0) = -9.8(0) + C, \qquad C = 30.$$

$$v(t) = -9.8t + 30.$$

b. $s'(t) = v(t) = -9.8t + 30, \;\; s(0) = 100$

$$s(t) = \int(-9.8t + 30)dt = -4.9t^2 + 30t + C$$

$$100 = s(0) = -4.9(0^2) + 30(0) + C, \quad C = 100.$$

$$s(t) = -4.9t^2 + 30t + 100.$$

c. Maximum height occurs when the velocity is 0.

$0 = -9.8t + 30$ or $t = \frac{30}{9.8} \approx 3.06s.$

$$s(3.06) = -4.9(3.06^2) + 30(3.06) + 100 \approx 145.92m.$$

d. We have to solve for the time when the rock hits the water.

$-4.9t^2 + 30t + 100 = 0$ (Use the quadratic formula)

$t = \frac{-30 \pm \sqrt{30^2 - 4(-4.9)(100)}}{2(-4.9)} \approx 8.52s$ (the other solution is negative)

$v(8.52) = -9.8(8.52) + 30 \approx -53.50m/s$ (– means downward)

Speed=$|velocity| = 53.30m/s.$