The Mean Value Theorem

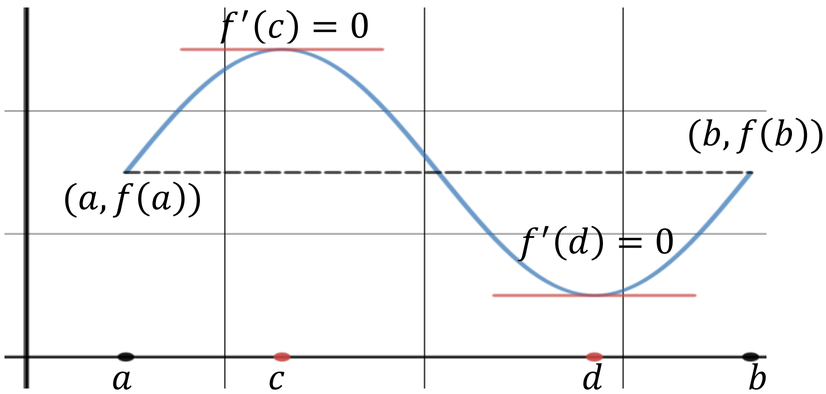
Rolle's Theorem: If

1. $f(x)$ is continuous on the closed interval $[a, b]$

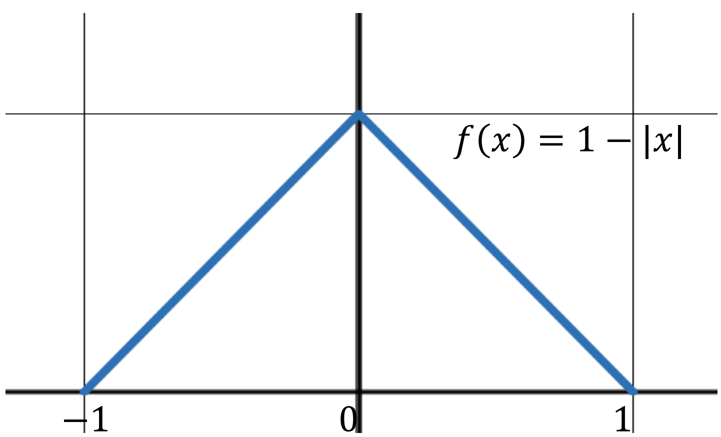
2. $f(x)$ is differentiable on the open interval (a, b)

3. $f(a) = f(b)$

Then there is at least one number c in (a, b) such the $f'(c) = 0$.



Ex. Notice that the function $f(x) = 1 - |x|$ on $[-1, 1]$ does not satisfy Rolle's theorem since it doesn't have a derivative at every point in $(-1, 1)$ (where doesn't it have a derivative?). If we draw the graph of $f(x) = 1 - |x|$ on $[-1, 1]$ we can see that there is no point where $f'(x) = 0$.



Ex. Verify that $f(x) = x^2 - 3x + 2$ satisfies Rolle's Thm on $[0,3]$ and find all values c that satisfy the conclusion of Rolle's Thm (ie, $f'(c) = 0$).

a. $f(x)$ is a polynomial so it is continuous everywhere. In particular, it's continuous on $[0,3]$.

b. $f(x)$ is a polynomial so it is differentiable everywhere. In particular, it's differentiable on $(0,3)$.

c. $f(0) = 2$, $f(3) = 3^2 - 3(3) + 2 = 2$. Thus $f(0) = f(3)$.

So $f(x)$ satisfies the conditions of Rolle's theorem.

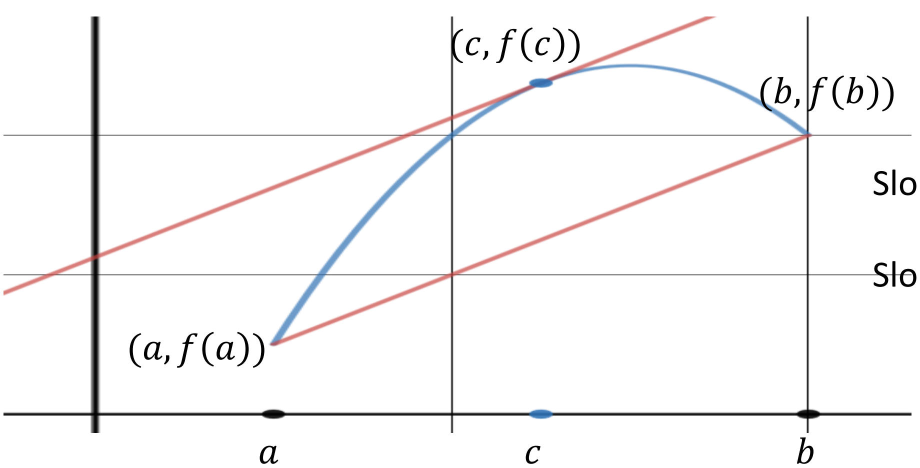
$$f'(x) = 2x - 3 = 0 \quad \Longrightarrow \quad x = \frac{3}{2}.$$

Thus $c = \frac{3}{2}$ is the only point in $[0,3]$ where $f'(x) = 0$.

The Mean Value Theorem: If

1. $f(x)$ is continuous on the closed interval $[a, b]$
2. $f(x)$ is differentiable on the open interval (a, b)

Then there is at least one number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Slope of secant line$= \frac{f(b)-f(a)}{b-a}$

Slope of tangent line$= f'(c)$.

Ex. Show $f(x) = x^3 - x$ satisfies the Mean Value Theorem (MVT) on $[0,2]$ and find all c's that satisfy the conclusion of the MVT.

a. $f(x)$ is a polynomial so it is continuous everywhere. In particular, it's continuous on $[0,2]$.

b. $f(x)$ is a polynomial so it is differentiable everywhere. In particular, it's differentiable on $(0,2)$.

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{[(2^3-2)-(0^3-0)]}{2} = 3.$$

$$f'(x) = 3x^2 - 1 \implies f'(c) = 3c^2 - 1$$

So $f'(c) = \frac{f(b)-f(a)}{b-a}$ when

$$3c^2 - 1 = 3$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3} \implies c = \pm\frac{2}{\sqrt{3}}$$

But only $c = \frac{2}{\sqrt{3}}$ is in the interval $(0,2)$.

Ex. Show $f(x) = \sqrt{x}$ satisfies the MVT on [1,9] and find all c's that satisfy the conclusion of the MVT.

a. $f(x)$ is continuous on [1,9] because it's a root function so it's continuous in its domain ($x > 0$).

b. $f(x)$ is differentiable on (1,9) because $f'(x) = \frac{1}{2\sqrt{x}}$ which exists in $(1,9)$.

$$\frac{f(b) - f(a)}{b - a} = \frac{\sqrt{9} - \sqrt{1}}{9 - 1} = \frac{3 - 1}{8} = \frac{1}{4}$$

$f'(x) = \frac{1}{2\sqrt{x}} \quad \Longrightarrow \quad f'(c) = \frac{1}{2\sqrt{c}}$

So $f'(c) = \frac{f(b)-f(a)}{b-a}$ when

$\frac{1}{2\sqrt{c}} = \frac{1}{4} \quad \Longrightarrow \quad 4 = 2\sqrt{c} \quad \Longrightarrow c = 4.$

Ex. Suppose a runner can run 21 miles in 3 hours. Assuming that the runner's speed is 0 at the start and finish, show that the runner must have been running at precisely 5 mph at least twice in the race (assume that the runner's position and velocity are differentiable functions on $(0, 21)$ and continuous on $[0, 21]$).

Notice that the runner's average velocity is $\frac{21}{3} = 7\ mph$. By the Mean Value Theorem $\frac{21}{3} = \frac{s(3.0)-s(0)}{3.0-0} = s'(c)$ for $0 < c < 3.0$. So the runner must have been running at 7 mph at some point. Since the runner's velocity is 0 at the beginning and end, by the intermediate value theorem, the runner must have been running at exactly 5 mph at least twice (once on $(0, c)$ and once on $(c, 3.0)$).

Theorem: If $f'(x) = 0$ for all x in (a, b), then $f(x)$ is a constant on (a, b).

Proof: We need to show that given any points x_1, x_2 with $a < x_1, x_2 < b$ that $f(x_1) = f(x_2)$.

Apply the Mean Value Theorem to the interval $[x_1, x_2]$:

$$0 = f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \Longrightarrow \quad f(x_2) = f(x_1).$$

Thus $f(x)$ is a constant on (a, b).

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then
$f(x) = g(x)$ +constant.

Proof: Let $h(x) = f(x) - g(x)$, then $h'(x) = 0$ in the interval (a, b), and thus by the previous theorem, $h(x) = f(x) - g(x) =$constant.

Thus $f(x) = g(x)$ +constant.

Theorem: Suppose $f(x)$ is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then $f(x)$ is increasing on I. If $f'(x) < 0$ at all interior points of I, then $f(x)$ is decreasing on I.

Proof when $f'(x) > 0$: Let x_1, x_2 be any points such that $a < x_1 < x_2 < b$ where a, b are endpoints of I. Applying the MVT to $[x_1, x_2]$ we get:

$$0 < f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \Longrightarrow \quad f(x_2) > f(x_1).$$

Thus $f(x)$ is increasing on I. The proof where $f'(x) < 0$ is similar.