Graphing Functions

Suggested steps for sketching a curve given a function $y = f(x)$.

1. Find the Domain: Identify what values of x are allowed in the function. For any polynomial this will be all real numbers.

2. Find intercepts: To find the y-intercept, plug $x = 0$ into the function. This is usually easy to calculate. The y-intercept is at $(0, f(0))$. The x-intercepts are calculated by solving for the x's where $f(x) = 0$. Sometimes this is hard to do, for example, $f(x) = 2x^3 - x^2 + x - 7$. If it's hard to find the x-intercepts then don't do it.

3. Find the asymptotes and determine end behavior: If $\lim_{n\to\infty} f(x)$ or $\lim_{n\to-\infty} f(x)$ equals L, then there is a horizontal asymptote at $y = L$. If $\lim_{x\to M^{\pm}} f(x) = \pm\infty$, then there is a vertical asymptote at $x = M$.

4. Take $f'(x)$ and find where its positive ($f(x)$ is increasing) and negative ($f(x)$ is decreasing). Identify any local maxima/minima by noting where $f'(x)$ changes sign (+ to -, or – to +) when going across points in the domain of $f(x)$.

5. Calculate $f''(x)$ and find where its positive ($f(x)$ is concave up) and negative ($f(x)$ is concave down). Identify any inflection points by noting where $f''(x)$ changes sign when going across points in the domain of $f(x)$.

6. Sketch the graph.

Ex. Sketch $(x) = \frac{2x^2}{x^2-1}$. Include the domain, intercepts, asymptotes, end behavior, where $f(x)$ is increasing/decreasing, local maxima/minima, where $f(x)$ is concave up/down, and any inflection points.

1. Domain: All real numbers except $x = \pm 1$.

2. Intercepts: $x = 0 \implies f(0) = \frac{2(0)^2}{(0)^2-1} = 0$, y-intercept is $(0,0)$.

$$y = 0 \implies \frac{2x^2}{x^2-1} = 0 \implies x = 0, \quad x\text{-intercept is } (0,0).$$

3. Asymptotes: Vertical asymptotes- $f(x) = \frac{2x^2}{x^2-1}$; $\quad x = \pm 1$.

   Horizontal asymptotes-

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{x^2(2)}{x^2(1-\frac{1}{x^2})} = 2 \quad \implies \quad y = 2.$$

4. Sign of $f'(x)$: Find where $f'(x)$ is 0 or undefined.

$$f'(x) = \frac{(x^2-1)4x - 2x^2(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} = 0 \quad \implies \quad x = 0.$$

$f'(x)$ is undefined at $x = \pm 1$.

Test the sign of $f'(x)$ on the intervals: $x < -1$, $\quad -1 < x < 0$, $0 < x < 1$, $\quad 1 < x$.

$$f'(-2) = \frac{-4(-2)}{((-2)^2-1)^2} = \frac{+}{+} = +$$

$$f'\left(-\frac{1}{2}\right) = \frac{-4(-\frac{1}{2})}{\left(\left(-\frac{1}{2}\right)^2-1\right)^2} = \frac{+}{+} = +$$

$$f'\left(\frac{1}{2}\right) = \frac{-4(\frac{1}{2})}{\left(\left(\frac{1}{2}\right)^2-1\right)^2} = \frac{-}{+} = -$$

$$f'(2) = \frac{-4(2)}{((2)^2-1)^2} = \frac{-}{+} = -$$

sign of $f'(x)$	$+$		$+$		$-$		$-$
		-1		0		1	

$f(x)$ is increasing for $x < -1$ or $-1 < x < 0$.

$f(x)$ is decreasing for $0 < x < 1$ or $1 < x$.

Relative maximum at $x = 0, \ \ y = 0$.

5. Sign of $f''(x)$: Find where $f''(x) = 0$ or is undefined.

$$f''(x) = -\left[\frac{\left(x^2-1\right)^2(4)-4x(2)\left(x^2-1\right)(2x)}{(x^2-1)^4}\right] = -[\frac{\left(x^2-1\right)\left(\left(x^2-1\right)(4)-16x^2\right)}{(x^2-1)^4}]$$

$$= -\left[\frac{-12x^2-4}{(x^2-1)^3}\right] = \frac{12x^2+4}{(x^2-1)^3}.$$

$f''(x) \neq 0$, since the numerator is always positive.

$f''(x)$ is undefined for $x = \pm 1$.

So we need to check the sign of $f''(x)$ on the intervals:

$$x < -1, \quad -1 < x < 1, \quad 1 < x.$$

$$f''(-2) = \frac{12(-2)^2+4}{((-2)^2-1)^3} = \frac{+}{+} = +$$

$$f''(0) = \frac{12(0)^2+4}{((0)^2-1)^3} = \frac{+}{-} = -$$

$$f''(2) = \frac{12(2)^2+4}{((2)^2-1)^3} = \frac{+}{+} = +$$

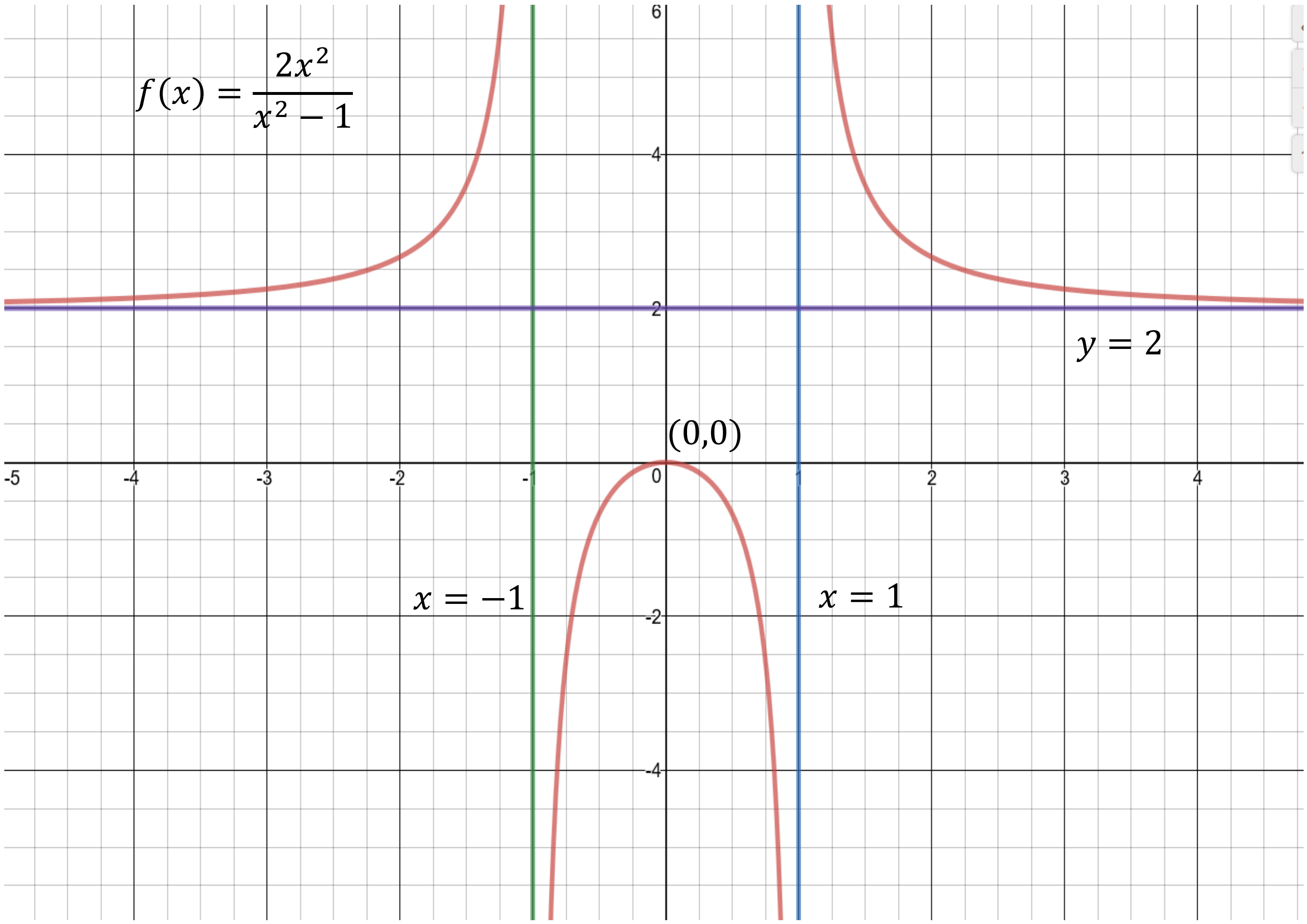
sign of $f''(x)$ _____+_____|_____$-$_____|____+____

-1 1

$f(x)$ is concave up when $x < -1$ or $1 < x$.

$f(x)$ is concave down when $-1 < x < 1$.

There are no inflection points because $x = \pm 1$ are not points of continuity.



Ex. Sketch a graph of $f(x) = \frac{x^2-9}{(x-1)^2}$ given that $f'(x) = \frac{2(9-x)}{(x-1)^3}$ and $f''(x) = \frac{4(x-13)}{(x-1)^4}$. Include all of the information required in the previous example.

1. Domain: All real numbers except $x = 1$.

2. Intercepts: $x = 0 \Longrightarrow f(0) = \frac{(0)^2-9}{(0-1)^2} = -9$, y-intercept is $(0, -9)$.

$$y = 0 \Longrightarrow \frac{x^2-9}{(x-1)^2} = 0 \Longrightarrow x = \pm 3, \quad x\text{-intercepts are } (\pm 3,0).$$

3. Asymptotes: Vertical asymptotes- $f(x) = \frac{x^2-9}{(x-1)^2}$; $x = 1.$

Horizontal asymptotes-

$$\lim_{x\to\pm\infty} \frac{x^2-9}{(x-1)^2} = \lim_{x\to\pm\infty} \frac{x^2(1-\frac{9}{x^2})}{x^2\left(1-\frac{1}{x}\right)^2} = 1 \quad \Longrightarrow \quad y = 1.$$

4. Sign of $f'(x)$: Find where $f'(x)$ is 0 or undefined.

$$f'(x) = \frac{2(9-x)}{(x-1)^3} = 0 \implies \quad x = 9.$$

$f'(x)$ is undefined at $x = 1$.

Test the sign of $f'(x)$ on the intervals: $x < 1$, $1 < x < 9$, $9 < x$.

$$f'(0) = \frac{2(9-0)}{(0-1)^3} = \frac{+}{-} = -$$

$$f'(2) = \frac{2(9-2)}{(2-1)^3} = \frac{+}{+} = +$$

$$f'(10) = \frac{2(9-10)}{(10-1)^3} = \frac{-}{+} = -$$

sign of $f'(x)$ ______ $-$ ______|______ $+$ ______|______ $-$ ______

1 9

$f(x)$ is increasing for $1 < x < 9$

$f(x)$ is decreasing for $x < -1$ or $9 < x$.

Relative maximum at $x = 9$, $y = \frac{(9)^2-9}{(9-1)^2} = \frac{72}{64} = \frac{9}{8}$.

Note: $x = 1$ is not a relative minimum because $x = 1$ is not a point of continuity.

5. Sign of $f''(x)$: Find where $f''(x) = 0$ or is undefined.

$$f''(x) = \frac{4(x-13)}{(x-1)^4} = 0 \quad \Longrightarrow \quad x = 13.$$

$f''(x)$ is undefined for $x = 1$.

So we need to check the sign of $f''(x)$ on the intervals:

$$x < 1, \quad 1 < x < 13, \quad 13 < x.$$

$$f''(0) = \frac{4(0-13)}{(0-1)^4} = \frac{-}{+} = -$$

$$f''(2) = \frac{4(2-13)}{(2-1)^4} = \frac{-}{+} = -$$

$$f''(14) = \frac{4(14-13)}{(14-1)^4} = \frac{+}{+} = +$$

sign of $f''(x)$ ______ $-$ ______|______ $-$ ______|____ $+$ ____

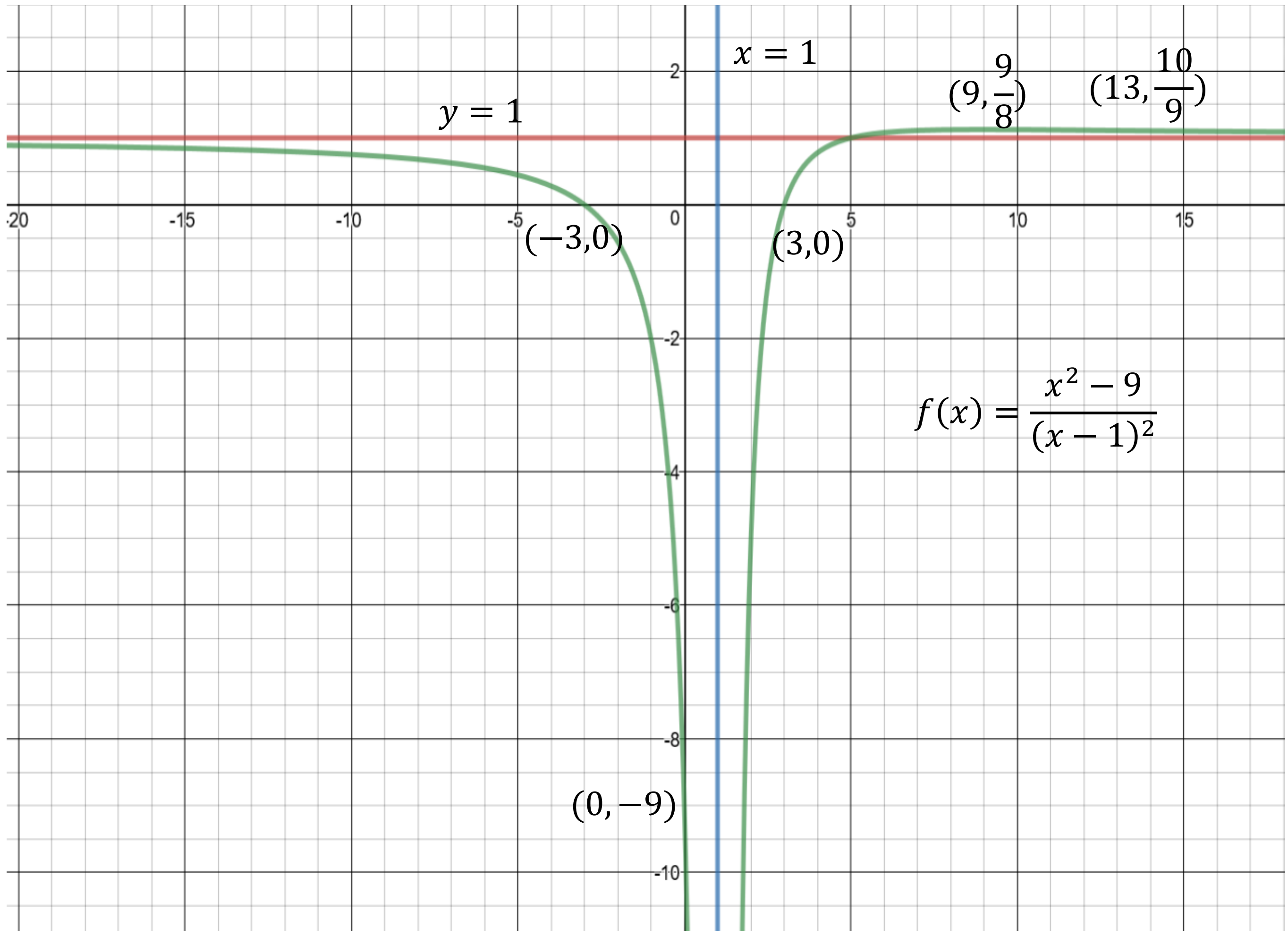
1 13

$f(x)$ is concave up when $13 < x$.

$f(x)$ is concave down when $x < 1$ or $1 < x < 13$.

There's an inflection point at $x = 13$, $y = \frac{(13)^2-9}{(13-1)^2} = \frac{160}{144} = \frac{13}{12}$.

$x = 1$ is not an inflection point because $x = 1$ is not a point of continuity and $f''(x)$ doesn't change sign.



$x = 1$
$(9, \frac{9}{8})$
$(13, \frac{10}{9})$
$y = 1$
2
-20
-15
-10
-5
0
5
10
15
$(-3,0)$
$(3,0)$
-2
$f(x) = \frac{x^2 - 9}{(x-1)^2}$
-4
-6
-8
$(0, -9)$
-10