The Concept of a Limit

Average Velocity$= \frac{\boldsymbol{Displacement}}{\boldsymbol{Time}}$.

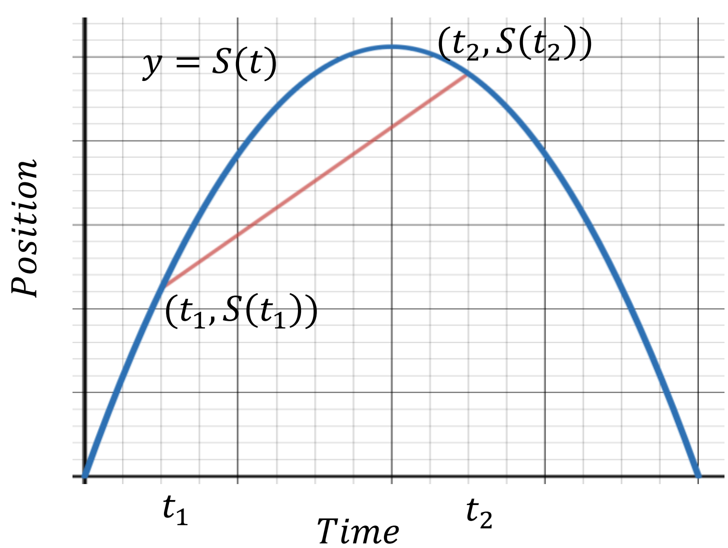
Ex. A driver leaves city A and drives 150 miles to city B. If the driver leaves city A at 2pm and arrives at city B at 5pm, find the average velocity.

Average Velocity$= \frac{150miles}{3hrs} = 50mph$

If the position of an object at time t is given by $S(t)$, then the average velocity for $t_1 \leq t \leq t_2$ is given by:

$$v_{ave} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

Notice that the average velocity is the slope of the secant line for $S(t)$ between t_1 and t_2.

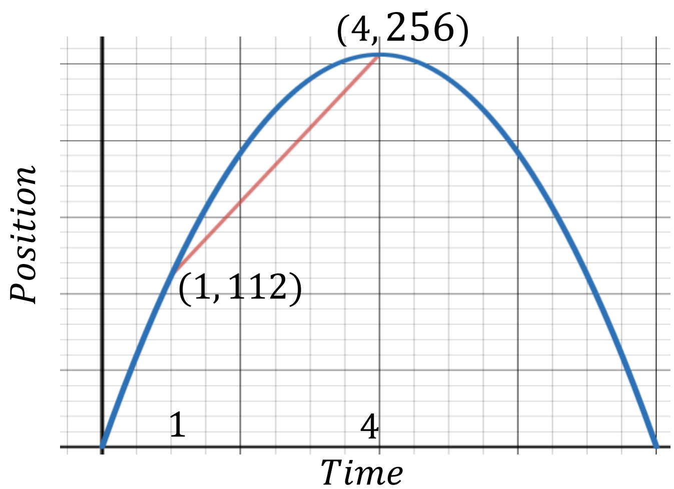


$$v_{ave} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}$$

=slope of secant line

Ex. A projectile is launched vertically upward at $128\, ft/sec$. Neglecting air resistance, the height of the projectile in feet above the ground after time $t \geq 0$ is given by $S(t) = -16t^2 + 128t, \quad 0 \leq t \leq 8\, sec.$

a. Find the average velocity between $t = 1\, sec$ and $t = 4\, sec.$
b. Find the average velocity between $t = 1\, sec$ and $t = 2\, sec.$
c. Find the average velocity between $t = 1\, sec$ and $t = 7\, sec.$



$$v_{ave} = \frac{S(4) - S(1)}{4 - 1}$$

a. For $1 \leq t \leq 4, \quad v_{ave} = \frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S(4) - S(1)}{4 - 1} = \frac{256 - 112}{4 - 1}$

$= 48\, ft/sec.$

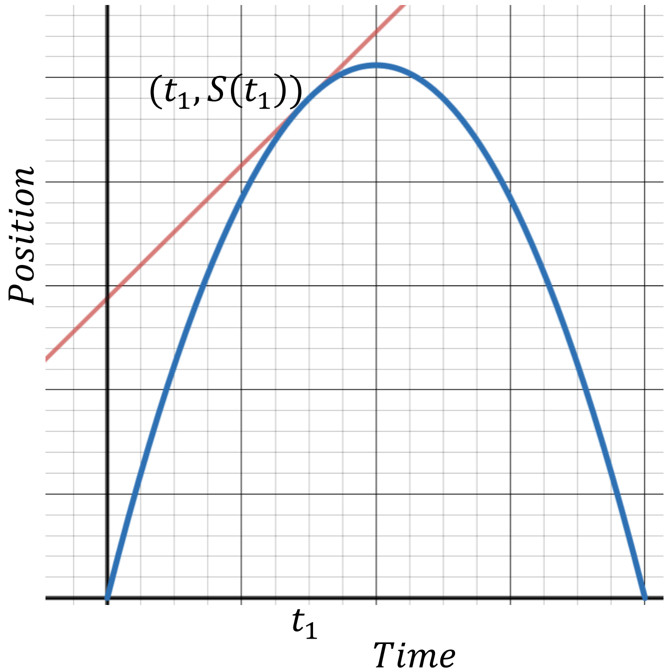
b. For $1 \leq t \leq 2, \quad v_{ave} = \frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S(2) - S(1)}{2 - 1} = \frac{192 - 112}{1}$

$= 80 ft/sec.$

c. For $1 \leq t \leq 7, \quad v_{ave} = \frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S(7) - S(1)}{7 - 1} = \frac{112 - 112}{4}$

$= 0\, ft/sec.$

We define the **Instantaneous Velocity** at t_1 to be the "limit" of the average velocities as t_2 approaches t_1. This will be the slope of the tangent line to $S(t)$ at t_1.



v_{inst} =slope of tangent line at $t = t_1$

$$\boxed{v_{inst} = \lim_{t \to t_1} \frac{S(t) - S(t_1)}{t - t_1}}$$

In the previous example we have:

For $1 \le t \le 4$, $\quad v_{ave} = 48\, ft/sec$

For $1 \le t \le 2$, $\quad v_{ave} = 80\, ft/sec$

For $1 \le t \le 1.1$, $\quad v_{ave} = \frac{S(1.1) - S(1)}{1.1 - 1} = 94.4\, ft/sec$

For $1 \le t \le 1.01$, $\quad v_{ave} = \frac{S(1.01) - S(1)}{1.01 - 1} = 95.84\, ft/sec.$

Approaching $t = 1$ from the left we get:

For $0 \leq t \leq 1$, $\quad v_{ave} = \frac{S(1)-S(0)}{1-0} = \frac{112-0}{1} = 112\ ft/sec$

For $0.9 \leq t \leq 1$, $\quad v_{ave} = \frac{S(1)-S(0.9)}{1-0.9} = \frac{112-102.24}{0.1} = 97.6\ ft/sec$

For $0.99 \leq t \leq 1$, $\quad v_{ave} = \frac{S(1)-S(0.99)}{1-0.99} = \frac{112-111.04}{0.01} = 96.16\ ft/sec$.

We will see that for this example

$$v_{inst} = m_{tan} = \lim_{t \to 1} \frac{S(t)-S(1)}{t-1} = 96\ ft/sec.$$

In general, if we want to find the average velocity on $[1, h]$, i.e., $1 \leq t \leq 1 + h$, we would get:

$$v_{ave} = \frac{S(1+h)-S(1)}{(1+h)-1} = \frac{-16(1+h)^2+128(1+h)-112}{h}$$

$$= \frac{-16\left(1+2h+h^2\right)+128+128h-112}{h}$$

$$= \frac{-32h-16h^2+128h}{h}$$

$$= \frac{96h-16h^2}{h}$$

$$= \frac{h(96-16h)}{h} = 96 - 16h.$$

If fact, this is the answer even if $h < 0$, i.e. $1 + h \leq t \leq 1$.

$$v_{inst} = m_{tan} = \lim_{t \to 1} \frac{S(t)-S(1)}{t-1} = \lim_{h \to 0} \frac{S(1+h)-S(1)}{(1+h)-1} = 96\ ft/sec.$$

There's nothing special about $t = 1\ sec$ in this example. We could ask for the instantaneous velocity for any time $t,\ 0 \le t \le 8\ sec$. For example, using $t = 2\ sec$ we get:

$$[2,3], \qquad v_{ave} = \frac{S(3)-S(2)}{3-2} = \frac{(240-192)}{1} = 48\ ft/sec$$

$$[2,2.1], \qquad v_{ave} = \frac{S(2.1)-S(2)}{2.1-2} = \frac{198.24-192}{0.1} = 62.4\ ft/sec$$

$$[2,2.01] \qquad v_{ave} = \frac{S(2.01)-S(2)}{2.01-2} = \frac{192.6384-192}{.01} = 63.84\ ft/sec$$

$$v_{inst} = m_{tan} = \lim_{t\to 2}\frac{S(t)-S(2)}{t-2} = 64\ ft/sec.$$

Ex. Find the average velocity in the previous example on the interval $[2, 2+h]$.

$$v_{ave} = \frac{S(2+h)-S(2)}{(2+h)-2} = \frac{-16(2+h)^2+128(2+h)-192}{h}$$

$$= \frac{-16\left(4+4h+h^2\right)+256+128h-192}{h}$$

$$= \frac{-64h-16h^2+128h}{h}$$

$$= \frac{64h-16h^2}{h}$$

$$= 64 - 16h.$$

Notice that:

$$v_{inst} = \lim_{t\to 2}\frac{S(t)-S(2)}{t-2} = \lim_{h\to 0}\frac{S(2+h)-S(2)}{(2+h)-2} = \lim_{h\to 0}(64-16h) = 64.$$

Ex. Find the slope of the secant line for $f(x) = -x^2 + 2$ on the interval:

a. $[2,3]$

b. $[1-h, 1]$

a. $m_{\text{sec}} = \frac{\mathrm{f(3)-f(2)}}{3-2} = \frac{\left(-(3^2)+2\right)-\left(-(2)^2+2\right)}{1}$

$= (-9+2) - (-4+2) = -7 - (-2) = -5.$

b. $m_{\text{sec}} = \frac{\mathrm{f(1)-f(1-h)}}{1-(1-\mathrm{h})} = \frac{\left(-(1)^2+2\right)-\left(-(1-h)^2+2\right)}{h}$

$= \frac{1-\left(-\left(1-2h+h^2\right)+2\right)}{h}$

$= \frac{1-\left(1+2h-h^2\right)}{h} = \frac{-2h+h^2}{h} = \frac{h(-2+h)}{h} = -2+h.$