Maximum and Minimum Values

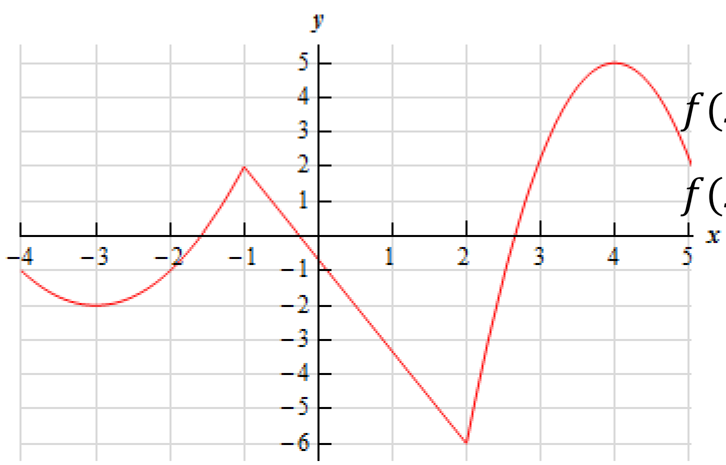
Def. A function $f(x)$ has an **Absolute Maximum** (or **Global Maximum**) at $x = c$ if

$f(c) \geq f(x)$ for all x in the domain of $f(x)$. $f(c)$ is called the **Maximum Value** of $f(x)$ for all x in the domain of $f(x)$.

A function $f(x)$ has an **Absolute Minimum** (or **Global Minimum**) at $x = c$ if

$f(c) \leq f(x)$ for all x in the domain of $f(x)$. $f(c)$ is called the **Minimum Value** of $f(x)$ for all x in the domain of $f(x)$.

The Maximum and Minimum Values of $f(x)$ are called the **Extreme Values** of $f(x)$.



$f(x)$ has a global maximum at $x = 4$.

$f(x)$ has a global minimum at $x = 2$.

Def. A function $f(x)$ has a **Local Maximum** (or **Relative Maximum**) at $x = c$ if

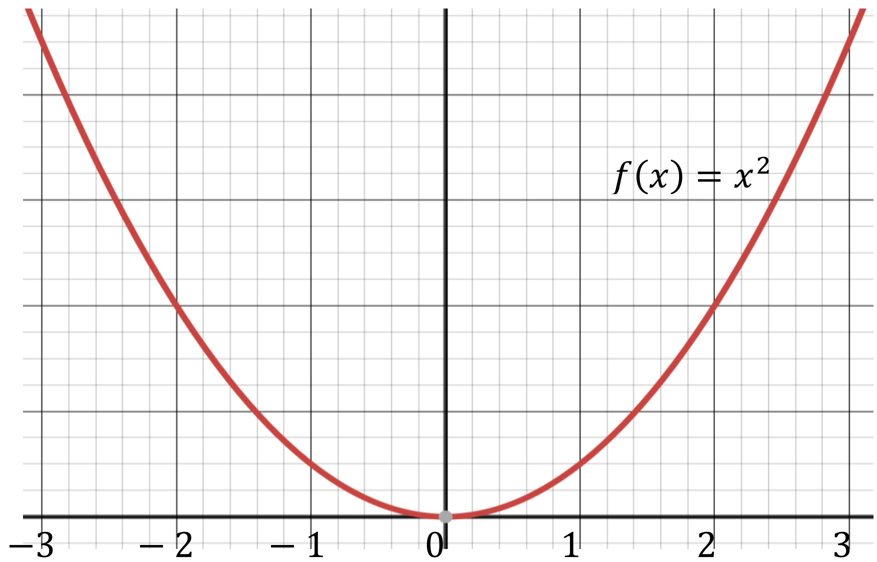
$f(c) \geq f(x)$ when x is near c.

A function $f(x)$ has a **Local Minimum** (or **Relative Minimum**) at $x = c$ if

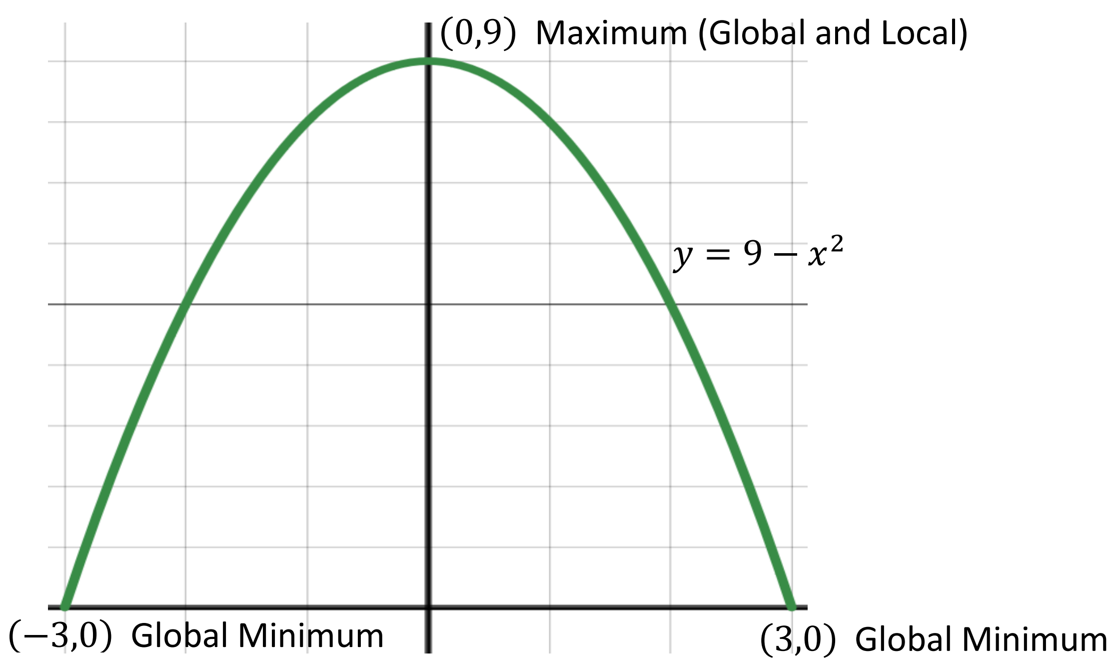
$f(c) \leq f(x)$ when x is near c.

(Note: $x = c$ cannot be an endpoint).

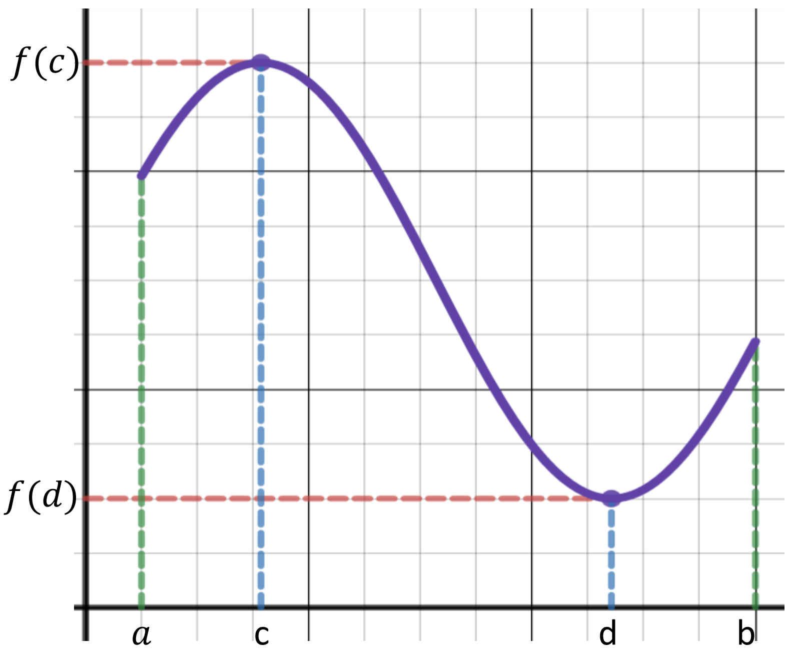
Ex. $f(x) = x^2$ for $-\infty < x < \infty$ has a local and global minimum at $x = 0$, but no global or local maximum.



Ex. $f(x) = 9 - x^2$ for $-3 \le x \le 3$ has a local and global maximum at $(0, 9)$ and global (but not local) minima at $(-3, 0)$ and $(3, 0)$.



Extreme Value Theorem: If $f(x)$ is a continuous function on a closed interval $a \leq x \leq b$, then $f(x)$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $a \leq x \leq b$.



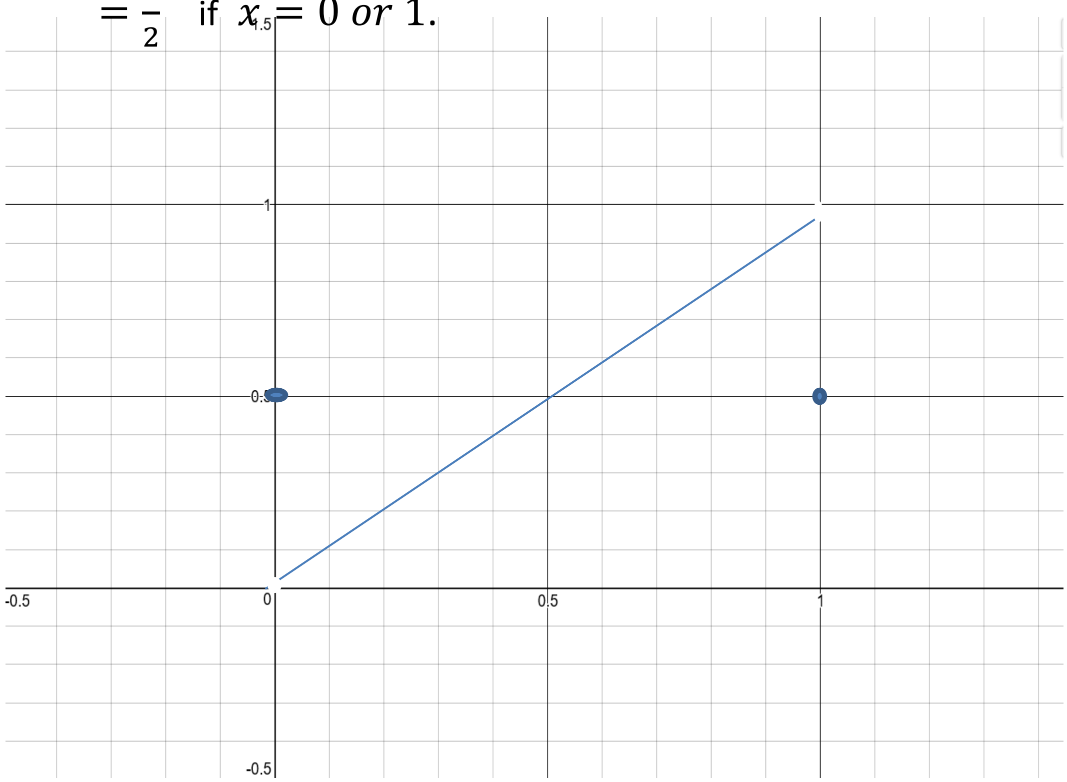
$f(x)$ has a global minimum at $x = d$.

$f(x)$ has a global maximum at $x = c$.

Note: if $f(x)$ is not continuous, or continuous but not on a closed interval, it need not (but could) take on its absolute maximum and minimum values.

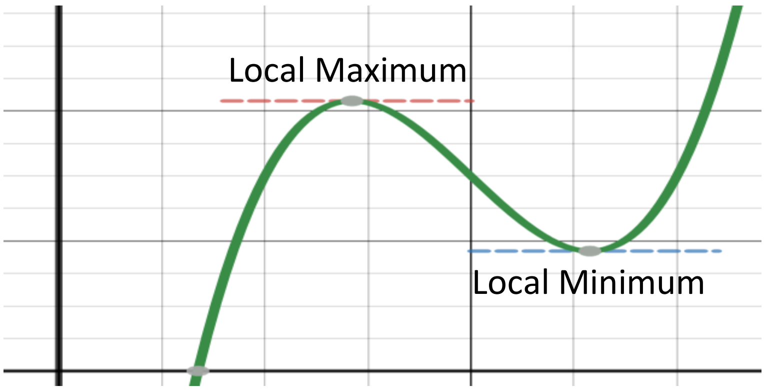
$f(x) = x$ if $0 < x < 1$ or $f(x) = x$ if $0 < x < 1$.

$= \frac{1}{2}$ if $x = 0 \; or \; 1.$

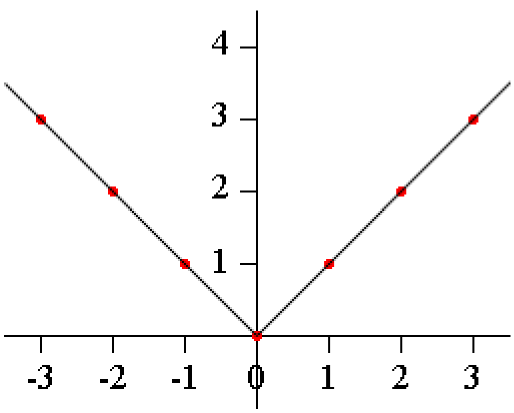


Given a function, $f(x)$, our goal is to find all local and global maxima/minima of $f(x)$.

Local Extreme Value Theorem: If $f(x)$ has a local maximum or minimum at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.

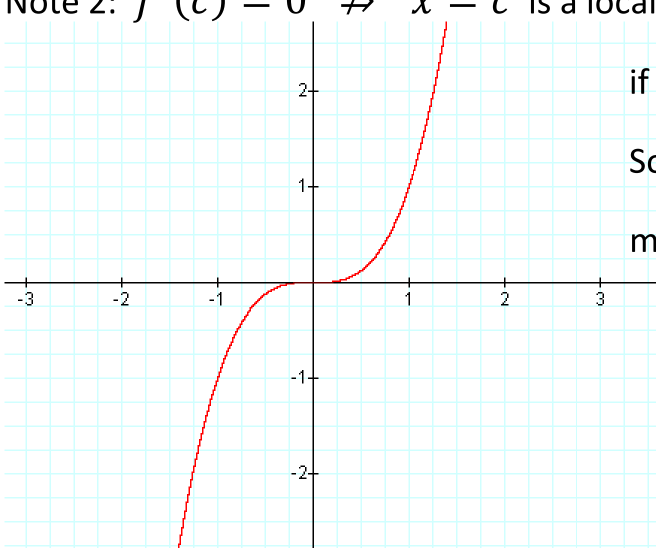


Note 1: It is possible to have a local maximum or minimum occur when $f'(c) \neq 0$ if $f'(c)$ is undefined at $x = c$.



$f(x) = |x|$ has a local minimum at $x = 0$, but $f'(0)$ is undefined.

Note 2: $f'(c) = 0 \not\Rightarrow x = c$ is a local maximum/minimum.



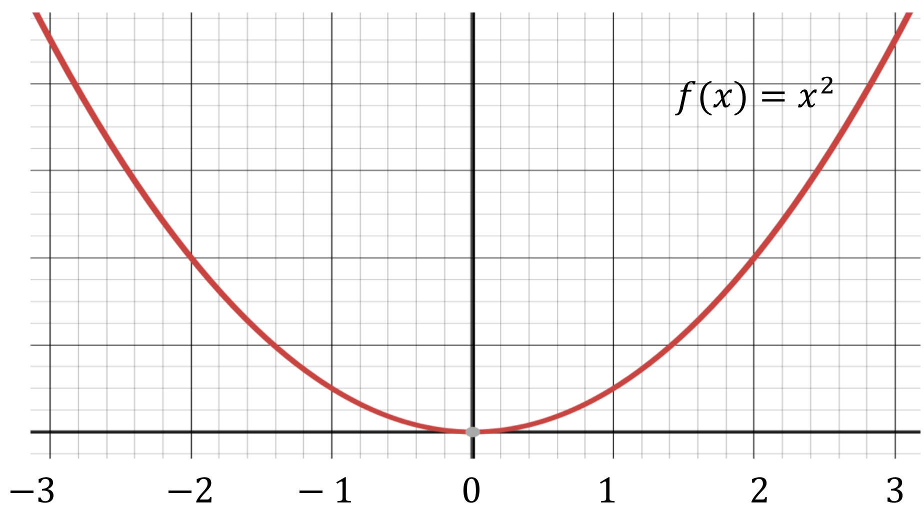
if $f(x) = x^3$ then $f'(x) = 3x^2$.

So $f'(0) = 0$, but $x = 0$ is not a local maximum/minimum.

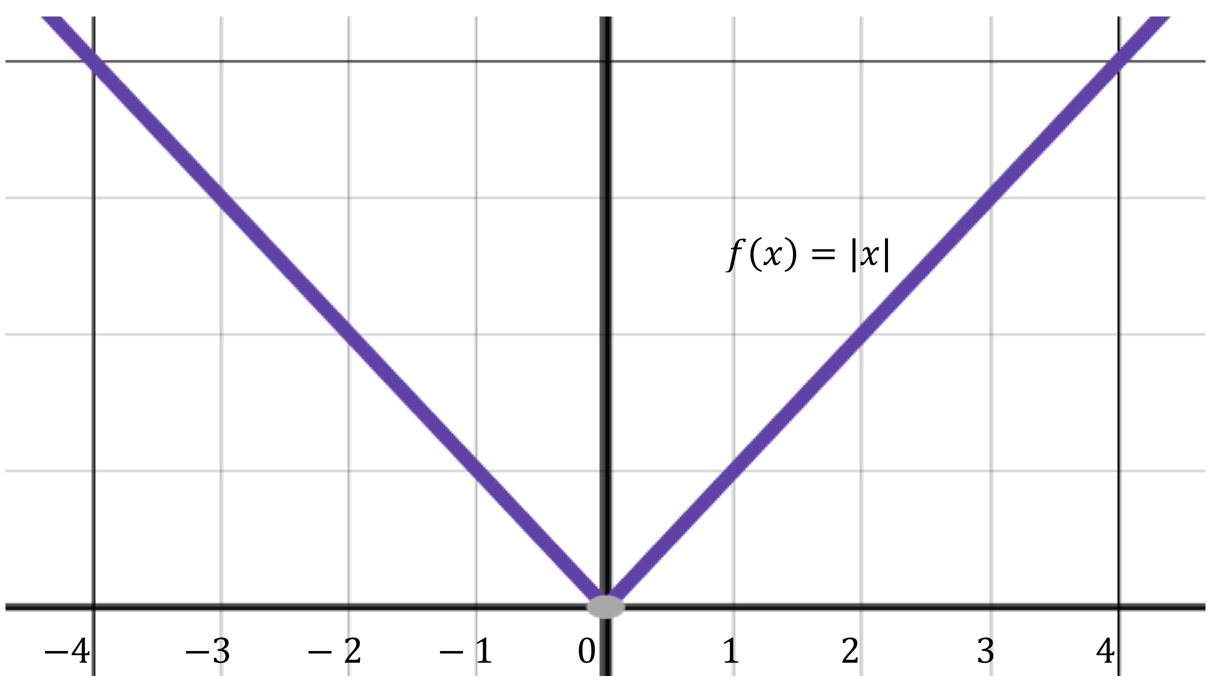
Def. A **Critical Point** or **Critical Number** of a function $f(x)$ is a number c in the domain of $f(x)$ such that $f'(c) = 0$ or $f'(c)$ doesn't exist.

Ex. Each of the following functions has a critical point at $x = 0$:

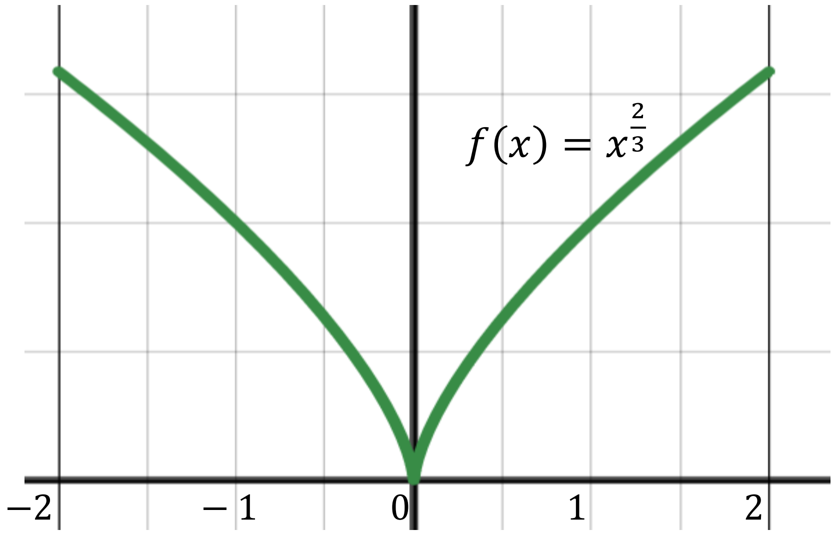
1. $f(x) = x^2$ because $f'(x) = 2x = 0$ when $x = 0$.



2. $f(x) = |x|$ because $f'(0)$ doesn't exist, but $x = 0$ is in the domain of f.



3. $f(x) = x^{\frac{2}{3}}$ because $f'(x) = \frac{2}{3x^{\frac{1}{3}}}$ so $f'(0)$ doesn't exist, but $x = 0$ is in the domain of $f(x)$.



Note: $f(x) = \frac{1}{x}$ does not have a critical point at $x = 0$ even though

$f'(x) = -\frac{1}{x^2}$ is undefined at $x = 0$ because $x = 0$ is not in the

domain of $f(x) = \frac{1}{x}$.

Theorem: If $f(x)$ has a local maximum or minimum at $x = c$, then $x = c$ is a critical point of $f(x)$.

Ex. Find the critical points for $f(x) = 2x^2 + 8x + 4$.

$f'(x) = 4x + 8 = 0 \quad \Longrightarrow \quad x = -2$ is the only critical number for $f(x)$.

Ex. Find the critical points for $f(x) = x^3 - 3x^2 - 9x + 1$.

$$f'(x) = 3x^2 - 6x - 9$$
$$= 3(x^2 - 2x - 3)$$
$$= 3(x - 3)(x + 1) = 0 \quad \Longrightarrow \quad x = 3, -1.$$

So the critical points are $x = 3, -1$.

Ex. Find the critical points for $f(x) = x - \frac{3}{2}x^{\frac{2}{3}}$.

$$f'(x) = 1 - x^{\left(-\frac{1}{3}\right)} = 0$$
$$1 = x^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{x}}$$

$\sqrt[3]{x} = 1 \quad \Longrightarrow \quad x = 1.$ So $x = 1$ is a critical point.

But $f'(x)$ is undefined at $x = 0$ and $x = 0$ is in the domain of $f(x)$,

so $x = 0$ is also a critical point.

Ex. Find the critical points for $f(x) = x + \frac{1}{x}$.

$$f'(x) = 1 - \frac{1}{x^2} = 0$$

$$1 = \frac{1}{x^2}$$

$$x^2 = 1 \implies x = \pm 1.$$

So $x = \pm 1$ are critical points.

$f'(x)$ is undefined at $x = 0$, but $x = 0$ is not in the domain of $f(x)$ so $x = 0$ is not a critical point.

Finding absolute maxima and minima on a closed interval

1. Find the values of $f(x)$ at the critical numbers of $f(x)$ in (a, b).
2. Find the values of $f(x)$ at the end points, $x = a$ and $x = b$.
3. The largest of the values from steps 1 and 2 is the absolute maximum. The smallest of the values from steps 1 and 2 is the absolute minimum.

Ex. Find the absolute maxima and minima for $f(x) = 2x^2 + 8x + 4$ for $-3 \leq x \leq 1$.

1. Find the values of $f(x)$ at the critical points of $f(x)$ for $-3 < x < 1$.

$f'(x) = 4x + 8 = 0 \quad \Longrightarrow \quad x = -2$ is the only critical point.

$f(-2) = 2(-2)^2 + 8(-2) + 4 = -4.$

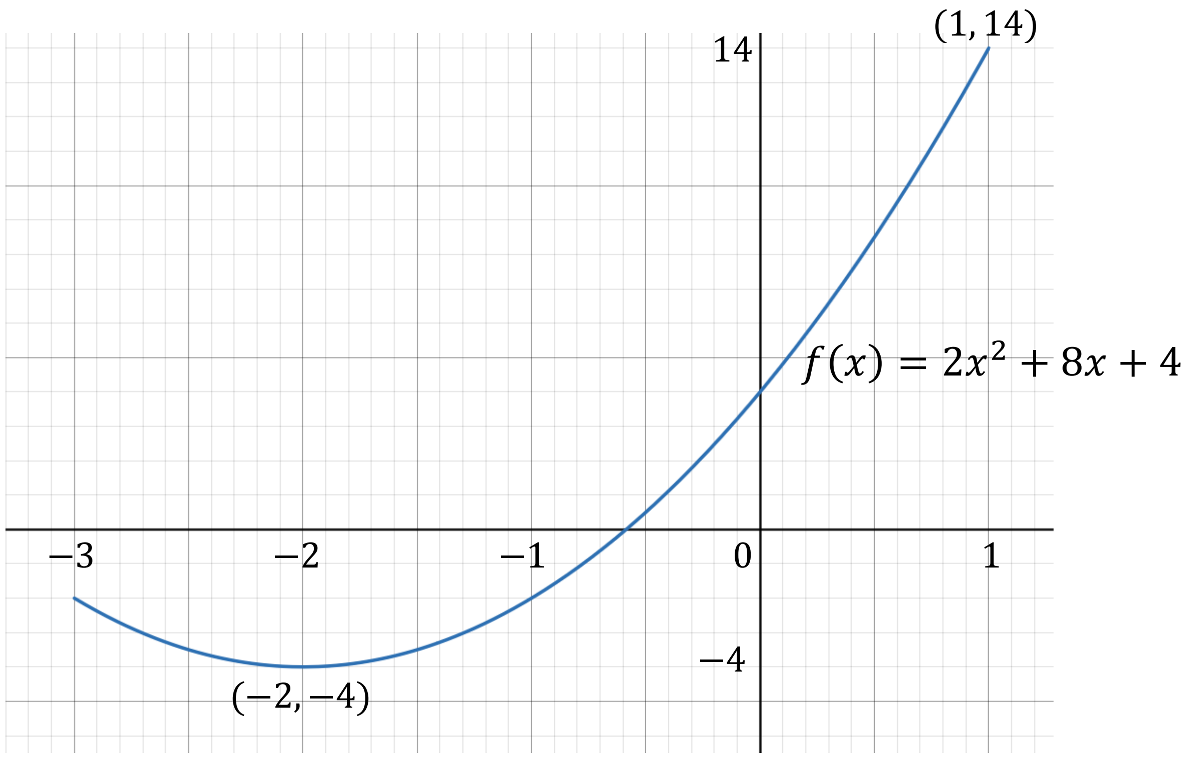
2. Find the values of $f(x)$ at the end points, $x = -3$ and $x = 1$.

$f(-3) = 2(-3)^2 + 8(-3) + 4 = -2$

$f(1) = 2(1)^2 + 8(1) + 4 = 14.$

3. Absolute maximum value$= 14$, which occurs at $x = 1$

Absolute minimum value$= -4$, which occurs at $x = -2$.



Ex. Find the absolute maximum and minimum values for $f(x) = x^3 + 3x^2$ for $-4 \leq x \leq \frac{1}{2}$.

1. Find the values of $f(x)$ at the critical points of $f(x)$ for $-4 < x < \frac{1}{2}$.

$$f'(x) = 3x^2 + 6x = 3x(x+2) = 0 \quad \Longrightarrow \quad x = 0, -2.$$

$$f(-2) = (-2)^3 + 3(-2)^2 = 4,$$

$$f(0) = 0.$$

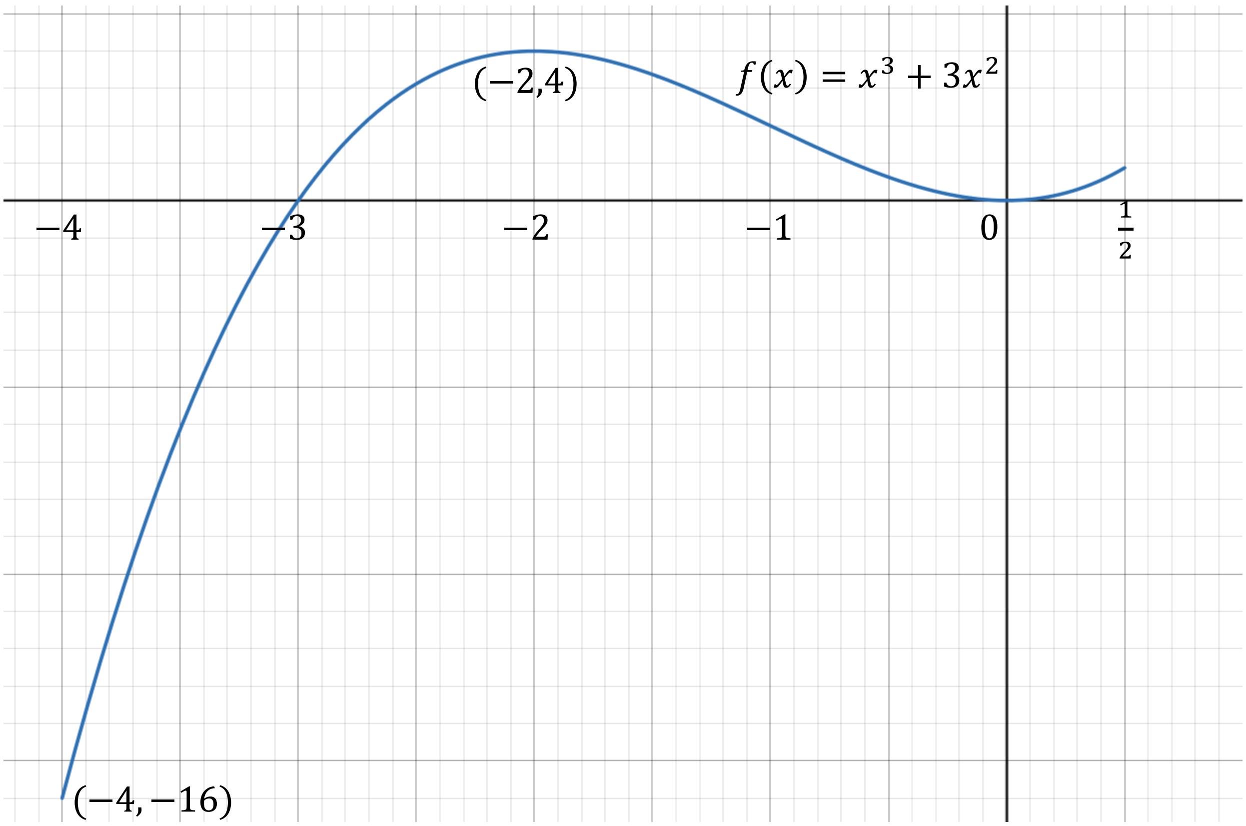
2. Find the values of $f(x)$ at the end points, $x = -4$ and $x = \frac{1}{2}$.

$$f(-4) = (-4)^3 + 3(-4)^2 = -16$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 = \frac{1}{8} + \frac{3}{4} = \frac{7}{8}.$$

3. Absolute maximum value $= 4$, which occurs at $x = -2$

   Absolute minimum value $= -16$, which occurs at $x = -4$.



Ex. Find the absolute maxima and minima for $f(x) = x - \sqrt{x}$ for $0 \leq x \leq 4$.

1. Find the values of $f(x)$ at the critical points of $f(x)$ for $0 < x < 4$.

$$f'(x) = 1 - \frac{1}{2\sqrt{x}} = 0$$

$$1 = \frac{1}{2\sqrt{x}}$$

$$2\sqrt{x} = 1$$

$$\sqrt{x} = \frac{1}{2}$$

$$x = \frac{1}{4}.$$

$$f\left(\frac{1}{4}\right) = \frac{1}{4} - \sqrt{\frac{1}{4}} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$$

Note: $x = 0$ is not a critical point because it's not in $0 < x < 4$.

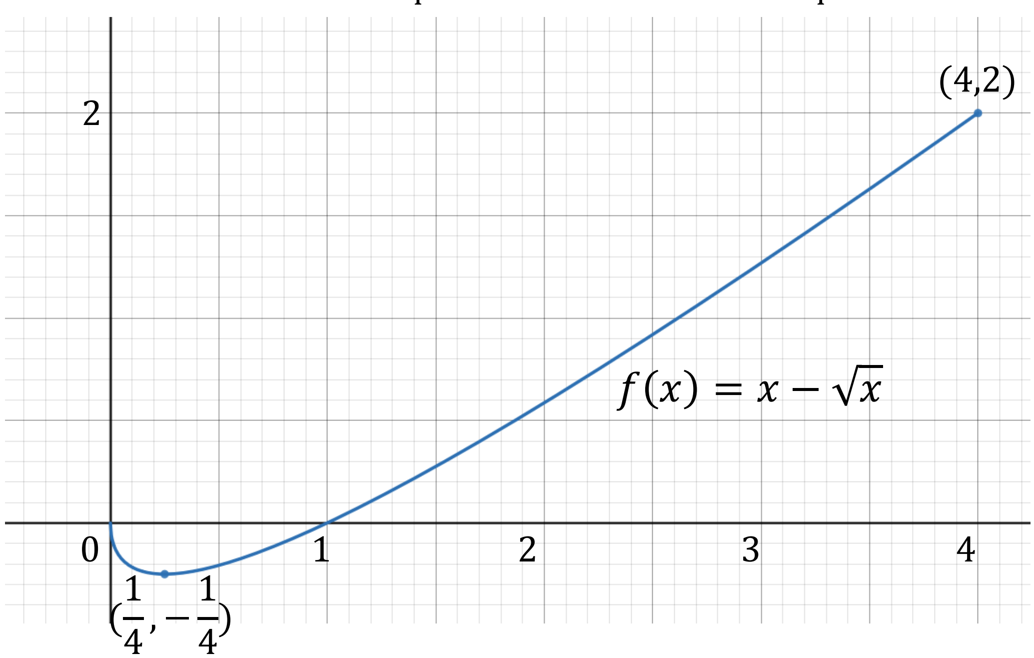
2. Find the values of $f(x)$ at the end points, $x = 0$ and $x = 4$.

   $f(0) = 0.$

   $f(4) = 4 - \sqrt{4} = 4 - 2 = 2.$

3. Absolute maximum value $= 2$, which occurs at $x = 4$

   Absolute minimum value $= -\frac{1}{4}$, which occurs at $x = \frac{1}{4}$.



Ex. Find the absolute maxima and minima for $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$;
$-1 \leq x \leq 8$.

1. Find the values of $f(x)$ at the critical points of $f(x)$ for $-1 < x < 8$.

$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}}$; critical points where $f'(x) = 0$ or undefined.

The trick to solving $5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 0$ is to factor out the lowest power, $x^{-\frac{1}{3}}$.

$$5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{-\frac{1}{3}}(x-2) = 0.$$

So $f'(x) = 0$ when $x = 2$ and $f'(x)$ is undefined at $x = 0$ (and $0 \in (-1,8)$)

So critical points at $x = 0, 2$.

$f(0) = 0, \qquad f(2) = 3(2)^{\frac{5}{3}} - 15(2)^{\frac{2}{3}} = (2)^{\frac{2}{3}}[3(2) - 15] = -9(2)^{\frac{2}{3}}.$

2. Find the values of $f(x)$ at the end points, $x = -1$ and $x = 8$.

$f(-1) = 3(-1)^{\frac{5}{3}} - 15(-1)^{\frac{2}{3}} = -3 - 15 = -18$

$$\begin{aligned} f(8) &= 3(8)^{\frac{5}{3}} - 15(8)^{\frac{2}{3}} \\ &= 3\left(\sqrt[3]{8}\right)^5 - 15\left(\sqrt[3]{8}\right)^2 \\ &= 3(2)^5 - 15(2)^2 \\ &= 96 - 60 = 36. \end{aligned}$$

3. Absolute maximum value= 36, which occurs at $x = 8$

   Absolute minimum value= -18, which occurs at $x = -1$.