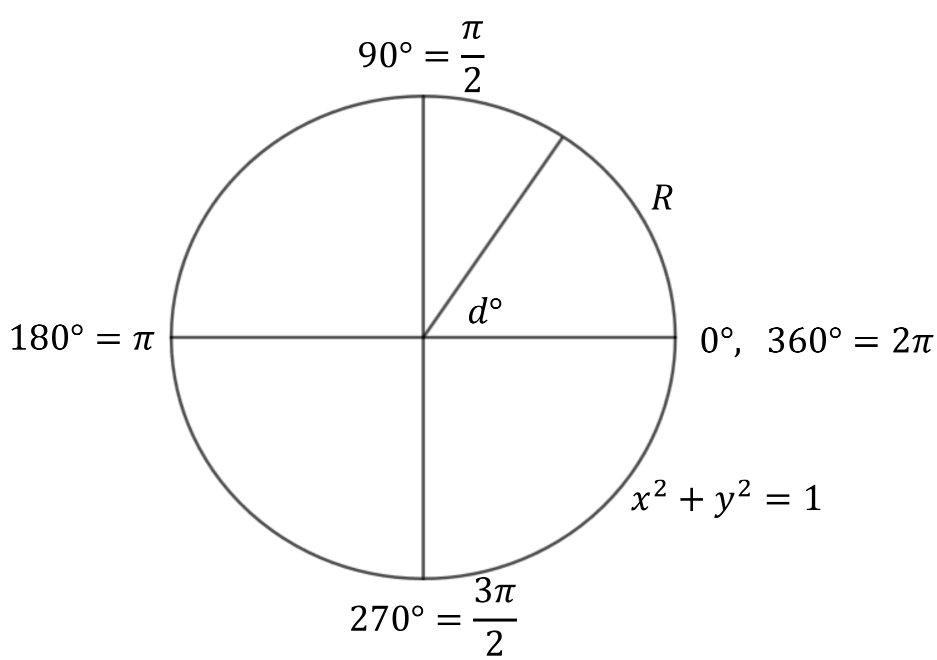
Trigonometric Functions

a. Converting Degrees to Radians

In Calculus you will be working in radians the vast majority of the time so you need to know how to convert degrees to radians (radians are real numbers).

The graph of $x^2 + y^2 = 1$ is a circle of radius 1 called "the unit circle". Its center is at $(0,0)$. The circumference of this circle is 2π. We associate the central angle of $d°$ with the length of the arc it makes, R, on the circle (see the diagram below). We say that d degrees equals R radians. Since the circumference of this unit circle is 2π, we say 360°=2π radians.



To convert degrees to radians, or radians to degrees, we use the formula:

$$\frac{\boldsymbol{d}}{\mathbf{360}} = \frac{\boldsymbol{R}}{\mathbf{2}\boldsymbol{\pi}}; \quad \text{where } d \text{ is in degrees and } R \text{ is in radians.}$$

Ex. Convert 90° to radians.

$$\frac{90}{360} = \frac{R}{2\pi}$$

$$\frac{1}{4} = \frac{R}{2\pi}$$

$$\frac{2\pi}{4} = R \quad \Rightarrow \quad \frac{\pi}{2} = R$$

So $90° = \frac{\pi}{2}$ radians.

Below are some common conversions you should memorize.

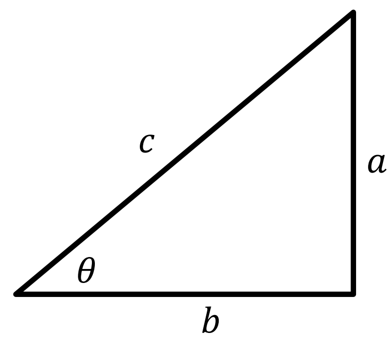
<u>Degrees</u>	<u>Radians</u>
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
180	π
270	$\frac{3\pi}{2}$
360	2π

b. Trigonometry of a Right Triangle

$\sin\theta = \frac{a}{c}$ $\csc\theta = \frac{c}{a}$

$\cos\theta = \frac{b}{c}$ $\sec\theta = \frac{c}{b}$

$\tan\theta = \frac{a}{b}$ $\cot\theta = \frac{b}{a}$

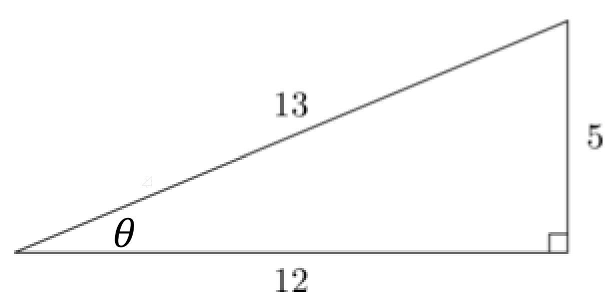


Ex. Find the value of the 6 trig functions for the following triangle

$\sin\theta = \frac{5}{13}$ $\csc\theta = \frac{13}{5}$

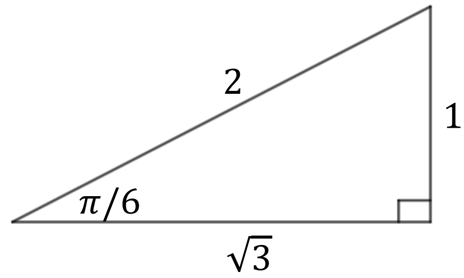
$\cos\theta = \frac{12}{13}$ $\sec\theta = \frac{13}{12}$

$\tan\theta = \frac{5}{12}$ $\cot\theta = \frac{12}{5}$.

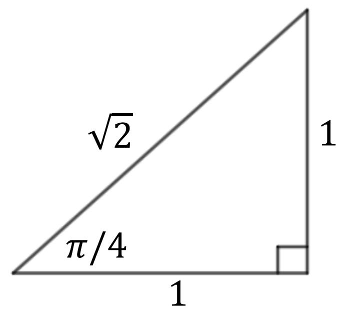


There are 3 "special" right triangles where we know the ratios of the lengths of the sides.

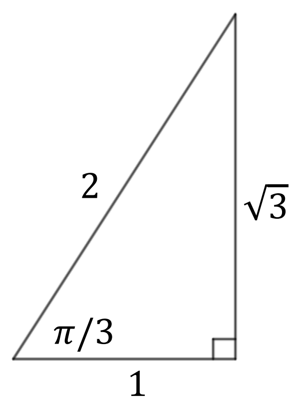
1. $\frac{\pi}{6} = 30°$



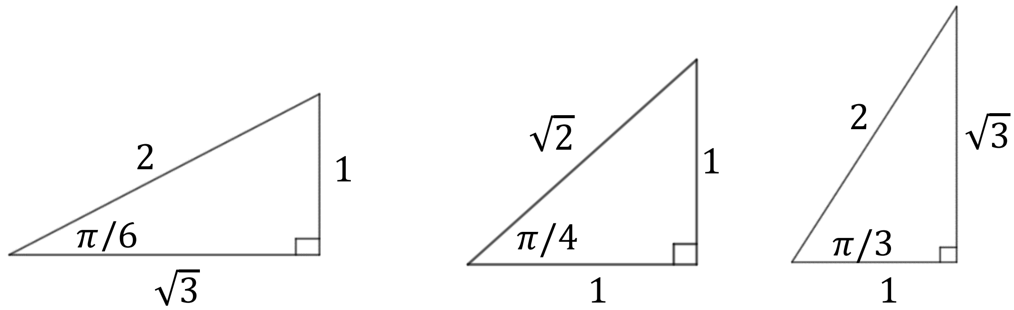
2. $\frac{\pi}{4} = 45°$



3. $\frac{\pi}{3} = 60°$



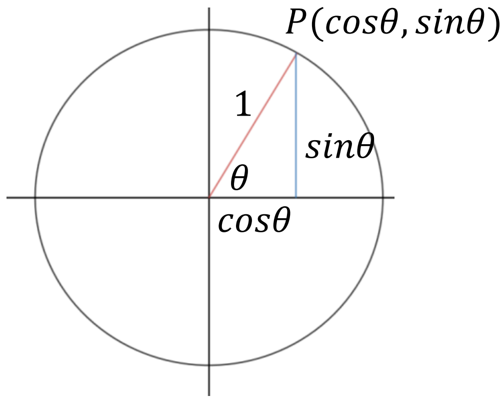
Using the definition of trig functions for a right triangle we can find the values of the 6 trig functions for these special triangles. You should either memorize this table or remember the ratio of the sides of the 3 special triangles so that you can reproduce this table. These values are going to come up again and again.



θ	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
$\csc\theta$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$
$\sec\theta$	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$.

c. Definitions of $\sin\theta$ and $\cos\theta$

If we draw a line segment from any point P on the circle $x^2 + y^2 = 1$ to $(0,0)$, it forms and angle θ with the x-axis (see diagram below). We define $\cos\theta$ to be the x-coordinate of P and $\sin\theta$ to be the y-coordinate of P.



Notice that this means we can now figure out the value of the sine and cosine for 0°=0 radians, $90° = \frac{\pi}{2}$ radians, 180°=π radians, and $270° = \frac{3\pi}{2}$ radians based on the coordinates of the point P corresponding to each of those points. Again, you need to memorize the values (or remember how to figure them out) of the sine and cosine of these numbers. As we will see, if you know the sine and cosine of a number, you can figure out the value of the other 4 trig functions from that.

Point P	Degrees	Radians	$\cos\theta$	$\sin\theta$
$(1,0)$	0	0	$cos0 = 1$	$sin0 = 0$
$(0,1)$	90	$\frac{\pi}{2}$	$cos\frac{\pi}{2} = 0$	$sin\frac{\pi}{2} = 1$
$(-1,0)$	180	π	$cos\,\pi = -1$	$sin\,\pi = 0$
$(0,-1)$	270	$\frac{3\pi}{2}$	$cos\frac{3\pi}{2} = 0$	$sin\frac{3\pi}{2} = -1.$

Also, since $(\cos\theta, \sin\theta)$ is a point on the unit circle we have:

$-1 \le \cos\theta \le 1$ and $-1 \le sin\theta \le 1.$

d. Definitions of the Other Four Trig Functions

We can now define the other four trig functions as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

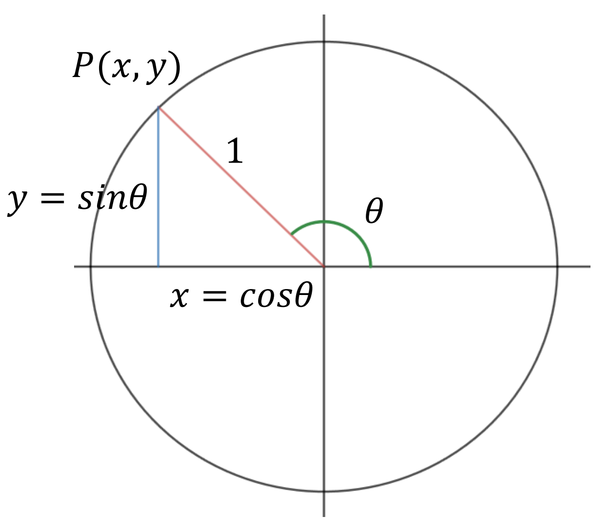
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}.$$

e. Where Various Trig Functions are Positive

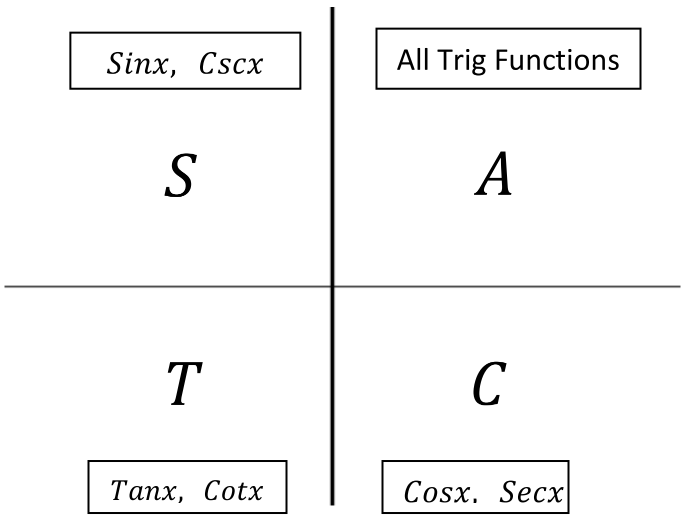
Notice that in each quadrant we have a different combination of signs of x and y. For example, in quadrant I, $x > 0, y > 0$, but in quadrant II, $x < 0, y > 0$. Since x=$\cos\theta$ and y= $sin\theta$ is a point on the unit circle, we can determine where the trig functions are positive and negative based on which quadrant θ is in.

Ex. θ is in the 2nd quadrant.



Quadrant of θ	sign of x	sign of y	Positive trig functions
I	+	+	all trig functions
II	–	+	sine and cosecant
III	–	–	tangent and cotangent
IV	+	–	cosine and secant.

In other words the trig functions are positive in the following quadrants:



There are many ways to remember this. I learned:

Always Study Trig Carefully.

Knowing where the trig functions are positive and negative allows us to know the value of the trig functions of numbers related to the special triangles. For example, if we want to know the $sin\,\frac{5\pi}{4}$, it will be the same as the $sin\,\frac{\pi}{4}$ except its sign may need to be adjusted based on what quadrant $\frac{5\pi}{4}$ is in. Since $\frac{5\pi}{4}$ is in the 3rd quadrant (between π and $\frac{3\pi}{2}$) and the sine is negative there, we have:

$$sin\,\frac{5\pi}{4} = -\,sin\,\frac{\pi}{4} = -\frac{\sqrt{2}}{2}.$$

Ex. Find $\frac{5\pi}{3}$.

$cos\,\frac{5\pi}{3}$ will have the same absolute value as $cos\,\frac{\pi}{3} = \frac{1}{2}$. The only question is if it's positive or negative. $\frac{5\pi}{3}$ is in the 4th quadrant (it's between $\frac{3\pi}{2}$ and 2π) and the cosine is positive there. Therefore, $cos\,\frac{5\pi}{3} = \frac{1}{2}$.

f. Some Important Trig Identities

The following 3 trig identities are called the Pythagorean trig identities.

Since $(\cos\theta, \sin\theta)$ is a point on $x^2 + y^2 = 1$, it must satisfy that equation,

1. $cos^2\theta + sin^2\theta = 1$ ($cos^2\theta$ means $(cos\theta)^2 = (cos\theta)(cos\theta)$)

This is the first of 3 Pythagorean trig identities. The other 2 come from this one. Here's how we get them:

Take $cos^2\theta + sin^2\theta = 1$ and divide the equation by $cos^2\theta$.

$$\frac{cos^2\theta}{cos^2\theta} + \frac{sin^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta}.$$

Using the fact that $\tan\theta = \frac{sin\theta}{cos\theta}$ and $sec\theta = \frac{1}{cos\theta}$, we get

2. $1 + tan^2\theta = sec^2\theta$

To get the 3rd Pythagorean identity, divide equation 1 by $sin^2\theta$.

$$\frac{cos^2\theta}{sin^2\theta} + \frac{sin^2\theta}{sin^2\theta} = \frac{1}{sin^2\theta}.$$

Using the fact that $\cot\theta = \frac{cos\theta}{sin\theta}$ and $\csc\theta = \frac{1}{sin\theta}$, we get

3. $cot^2\theta + 1 = csc^2\theta$

4. Some other useful Trig formulas:

$sin2\theta = 2cos\theta sin\theta$

$cos2\theta = cos^2\theta - sin^2\theta = 1 - 2sin^2\theta = 2cos^2\theta - 1$

$sin^2\theta = \frac{1-cos2\theta}{2}$

$cos^2\theta = \frac{1+cos2\theta}{2}$.

5. The following formula (the sine of the sum of two numbers) is used when deriving the formula for the derivative of sin$x$:

$$\sin(x+h) = \sin x\,(\cos(\mathrm{h})) + \sin h\,(\cos(x))$$

$$\cos(x+h) = cosx(\cos(h)) - sinx(\sin(h)).$$

g. Solving Trig Equations

Solving trig equations often shows up when you are trying to find the maximum or minimum of a function that involves trig functions. The hardest part of solving trig equations is remembering to get <u>all</u> of the solutions. This usually entails knowing the values of the trig functions for special angles, remembering that there may be solutions in quadrants other than the 1st quadrant, and that every 2π you may get another solution (depending on the restriction on the solutions).

Ex. Solve $2cosx - \sqrt{3} = 0$.

First we solve for $cosx$.

$$2cosx = \sqrt{3}$$

$$cosx = \frac{\sqrt{3}}{2}$$

Now the question is, what numbers, x, have their cosine equal to $\frac{\sqrt{3}}{2}$? Here's a case where it's helpful to know the trig values of the special angles.

One solution is $x = \frac{\pi}{6}$, however, there are many more. This is just the answer in the first quadrant.

We also know that the cosine is positive in the 4th quadrant. So what angle related to $\frac{\pi}{6}$ (i.e. it's a multiple of $\frac{\pi}{6}$, the fraction can't be simplified, and it's in the 4th quadrant)? The answer is $\frac{11\pi}{6}$.

So $\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$; meaning $x = \frac{11\pi}{6}$ is also a solution.

But there are more solutions. Every time we add a multiple of 2π to each of our answers, we get other answers. So the complete solution is:

$x = \frac{\pi}{6} + 2n\pi$ and $x = \frac{11\pi}{6} + 2n\pi$, where n is any integer.

Notice that there wasn't any restriction on the solution, x, in the original problem. If the problem had been:

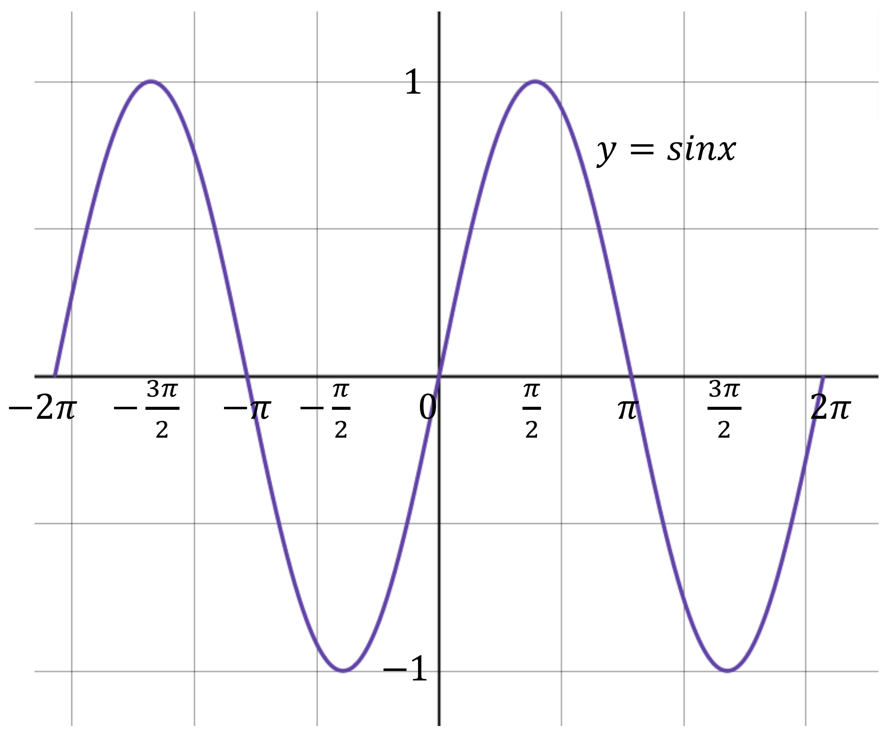
Solve $2cosx - \sqrt{3} = 0$; $0 \leq x \leq 2\pi$

Our only answers would have been:

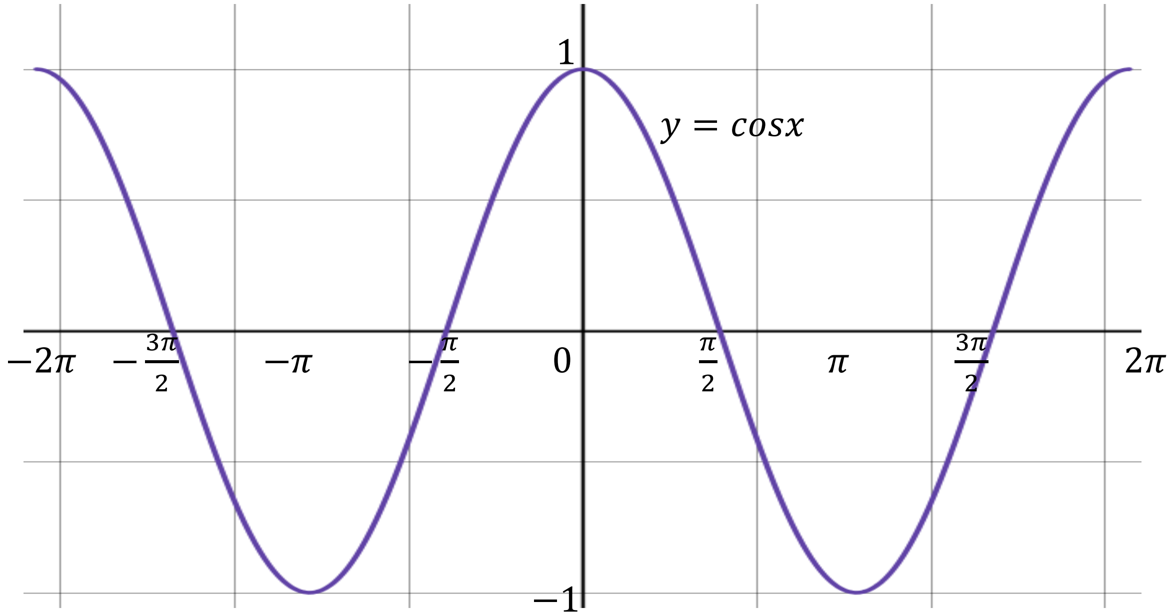
$x = \frac{\pi}{6}, \frac{11\pi}{6}$.

g. The Graphs of the Sine, Cosine, and Tangent

$$y = sinx; \quad period = 2\pi, \quad Domain = all\ real\ \#s, \quad Range: -1 \leq y \leq 1$$



$y = cosx; \quad period = 2\pi, \;\; Domain = all\ real\ \#s, \;\; Range: -1 \le y \le 1$



$y = tanx; \;\; period = \pi,$

$$Domain = all\ real\ \#s\ except\ \frac{\pi}{2} + n\pi, Range: -\infty < y < \infty.$$